
Geometry of Quantum Projective Spaces

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1 Introduction

In recent years, several quantizations of real manifolds have been studied, in particular from the point of view of Connes' noncommutative geometry [7]. Less is known for complex noncommutative spaces. A natural first step in developing a theory is clearly the study of quantizations of flag manifolds — and, in particular, complex projective spaces —, thus explaining the increasing interest for this class of examples. In this paper, we review various aspects of the geometry of deformations of complex projective spaces.

Some references on these topics are the following. For Fredholm modules and classical characteristic classes, as well as equivariant K-theory and quantum characteristic classes, one can see [36, 21, 22]; differential calculi have been studied by several authors, e.g. [6, 56, 41, 34, 35, 3, 4]; for Dirac operators and spectral triples we refer to [25, 41, 18, 19, 21]; complex structures and positive cyclic cocycles have been studied in [37, 38, 39]; for monopoles and instantons in the 4-dimensional case, we refer to [22, 23]. The quantum projective line has been also used as the “internal space” for a scheme of equivariant dimensional reduction leading to q -deformations of systems of non-abelian vortices in [44]. In the (complex) 1-dimensional case, in [55] there is a study of some of the “seven axioms” of noncommutative geometry.

An original part of the present work is a proof of rational Poincaré duality for a new family of real spectral triples, generalizing the one in [55]. Other original results include: the computation in §7.2 of the cohomology of the Dolbeault complex of \mathbb{CP}_q^n ; in §7.3 we give in full details an easier and complete proof of a dimension formula (Cor. 4.2 of [39]) for the zero-th cohomology of holomorphic modules; in §7.4 we exhibit $n + 1$ positive Hochschild twisted cocycles that, in §7.5 we pair with equivariant K-theory, thus also showing the pairwise inequivalence of the projections in Prop. 3.1.

Notations. We shall have $0 < q \leq 1$ as deformation parameter, with $q = 1$ corresponding to the “classical limit”. The q -analogue of an integer number n is defined as $[n] := (q^n - q^{-n})/(q - q^{-1})$ for $q \neq 1$ and equals n in the limit $q \rightarrow 1$. For any $n \geq 1$, the q -factorial is $[n]! := [n][n - 1] \dots [1]$, with $[0]! := 1$, and, for j_0, \dots, j_n interger numbers, the q -multinomial coefficients is

$$[j_0, \dots, j_n]! := \frac{[j_0 + \dots + j_n]!}{[j_0]! \dots [j_n]!}.$$

By $*$ -algebra we shall always mean an unital involutive associative algebra over the complex numbers, and by representation of a $*$ -algebra we always

mean a unital $*$ -representation, unless otherwise stated. For a coproduct we use Sweedler notation, $\Delta(x) = x_{(1)} \otimes x_{(2)}$, with summation understood.

2 The quantum $SU(n+1)$ and \mathbb{CP}^n

2.1 Quantized ‘coordinate rings’

In the framework of C^* -algebras, the compact quantum groups $SU_q(n)$, for $n \geq 2$, have been introduced in [60]. It is well known that any compact quantum group has a dense subalgebra which is a Hopf $*$ -algebra with the induced coproduct. This Hopf $*$ -algebra is the analogue of the algebra of representative functions of a compact group. For $SU_q(n)$ it will be denoted by $\mathcal{A}(SU_q(n))$; it has been studied in [29] (among others) and the exact definition can be found for example in [40, Sect. 9.2]. Here we recall that, for any $n_1 > n_2$, there is a surjective Hopf $*$ -algebra morphism $\mathcal{A}(SU_q(n_1)) \rightarrow \mathcal{A}(SU_q(n_2))$ or, in other words, $SU_q(n_2)$ is a “quantum subgroup” of $SU_q(n_1)$.

For $n \geq 1$, the “quotient” $SU_q(n+1)/SU_q(n)$ leads to the so-called odd-dimensional quantum spheres S_q^{2n+1} , the natural ambient space when studying quantum projective spaces. More precisely, the coordinate algebra $\mathcal{A}(S_q^{2n+1})$ is defined as the $*$ -subalgebra of $\mathcal{A}(SU_q(n+1))$ made of coinvariant elements for the coaction of $SU_q(n)$. As an abstract $*$ -algebra, this is generated by $2(n+1)$ elements $\{z_i, z_i^*\}_{i=0}^n$ with commutation relations [54]:

$$\begin{aligned} z_i z_j &= q^{-1} z_j z_i, \quad 0 \leq i < j \leq n \quad \text{and} \quad z_i^* z_j = q z_j z_i^*, \quad i \neq j, \\ [z_n^*, z_n] &= 0 \quad \text{and} \quad [z_i^*, z_i] = (1 - q^2) \sum_{j=i+1}^n z_j z_j^*, \quad i = 0, \dots, n-1, \end{aligned}$$

and sphere condition:

$$z_0 z_0^* + z_1 z_1^* + \dots + z_n z_n^* = 1.$$

Using the commutation relations above, an equivalent way to write the sphere condition is $\sum_{j=0}^n q^{2j} z_j^* z_j = 1$. In the case $n = 1$, one has $SU_q(1) = \{1\}$ and the quantum sphere S_q^3 is the ‘manifold’ underlying the quantum $SU(2)$ group. The generator $z_0 = \alpha$ and $z_1 = \beta$ of the algebra $\mathcal{A}(S_q^3)$ can be assembled into a matrix

$$U := \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix}.$$

The defining relations are then encoded in the condition $UU^* = U^*U = I_2$, where I_2 is the 2×2 identity matrix. With standard ‘matrix’ coproduct, counit and antipode, this gives the well known quantum group $SU_q(2)$ of [58, 59].

The original notations of [54] are obtained by setting $q = e^{\hbar/2}$; the generators of [19] correspond to the replacement $z_i \rightarrow z_{n+1-i}$, while the generators x_i of [33] are related to ours by $x_i = z_{n+1-i}^*$ and by the replacement $q \rightarrow q^{-1}$.

For any $n \geq 1$, the $*$ -subalgebra of $\mathcal{A}(S_q^{2n+1})$ generated by $p_{ij} := z_i^* z_j$ will be denoted $\mathcal{A}(\mathbb{CP}_q^n)$, and called the algebra of ‘polynomial functions’ on the quantum projective space \mathbb{CP}_q^n . The algebra $\mathcal{A}(\mathbb{CP}_q^n)$ is made of invariant elements for the $U(1)$ action $z_i \rightarrow \lambda z_i$ for $\lambda \in U(1)$.

From the relations of $\mathcal{A}(S_q^{2n+1})$ one gets analogous quadratic relations for $\mathcal{A}(\mathbb{CP}_q^n)$ [21]. In particular, the elements p_{ij} are the matrix entries of a projection $P = (p_{ij})$, i.e. $P^2 = P = P^*$ or $\sum_{j=0}^n p_{ij} p_{jk} = p_{ik}$ and $p_{ij}^* = p_{ji}$. This projection has q -trace:

$$\mathrm{Tr}_q(P) := \sum_{i=0}^n q^{2i} p_{ii} = 1. \quad (1)$$

For $n = 1$, \mathbb{CP}_q^1 is also a deformation of the unit sphere S^2 known as the “standard” Podleś quantum sphere [51].

A further generalization is given by quantum weighted projective spaces $\mathbb{WP}_q(k_0, k_1, \dots, k_n)$, where $\{k_i\}_{i=0}^n$ are pairwise coprime numbers. The corresponding coordinate algebra is the fixed point subalgebra of $\mathcal{A}(S_q^{2n+1})$ for the action $z_i \mapsto \lambda^{k_i} z_i$ of $U(1)$. For $n = 1$, these are called “quantum teardrops” and studied in [5]. Their discussion is beyond the scope of this review.

2.2 Symmetry algebras

Let \mathcal{A} be a $*$ -algebra, $(\mathcal{U}, \epsilon, \Delta, S)$ a Hopf $*$ -algebra. One says that \mathcal{A} is a left \mathcal{U} -module $*$ -algebra if there is a left action ‘ \triangleright ’ of \mathcal{U} on \mathcal{A} such that

$$x \triangleright ab = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b), \quad x \triangleright 1 = \epsilon(x)1, \quad x \triangleright a^* = (S(x)^* \triangleright a)^*,$$

for all $x \in \mathcal{U}$, $a, b \in \mathcal{A}$. With this data, one defines the left crossed product algebra $\mathcal{A} \rtimes \mathcal{U}$. i.e. the $*$ -algebra generated by \mathcal{A} and \mathcal{U} with crossed commutation relations $xa = (x_{(1)} \triangleright a)x_{(2)}$, for all $x \in \mathcal{U}$ and $a \in \mathcal{A}$. There are analogous notions of a right \mathcal{U} -module $*$ -algebra and right crossed product algebra.

Symmetries of $SU_q(n+1)$ and of related quotient spaces are described by the action of the dual Hopf $*$ -algebra, here denoted by $U_q(\mathfrak{su}(n+1))$. This is

the ‘compact’ real form of the Hopf algebra denoted $\check{U}_q(\mathfrak{sl}(n+1, \mathbb{C}))$ in §6.1.2 of [40]. Left and right canonical (and commuting) actions of $U_q(\mathfrak{su}(n+1))$ on $\mathcal{A}(SU_q(n+1))$ will be denoted by \triangleright and \triangleleft respectively.

If $h : \mathcal{A}(SU_q(n+1)) \rightarrow \mathbb{C}$ is the Haar state, i.e. the unique invariant state on the algebra, a inner product is defined as usual by $\langle a, b \rangle := h(a^*b)$. It turns out that the left action is unitary for this inner product, that is $\langle a, x \triangleright b \rangle = \langle x^* \triangleright a, b \rangle$ for all $a, b \in \mathcal{A}(SU_q(n+1))$ and $x \in U_q(\mathfrak{su}(n+1))$. The right action is not unitary, but it can be turned into a second unitary left action \mathcal{L} , commuting with the former one, via the rule

$$\mathcal{L}_x a := a \triangleleft S^{-1}(x) .$$

The algebra $\mathcal{A}(S_q^{2n+1})$ can be identified with the $*$ -subalgebra of $\mathcal{A}(SU_q(n+1))$ fixed by the \mathcal{L} -action of the Hopf $*$ -subalgebra $U_q(\mathfrak{su}(n)) \subset U_q(\mathfrak{su}(n+1))$; whereas the projective space $\mathcal{A}(\mathbb{CP}_q^n)$ is the $*$ -subalgebra of $\mathcal{A}(S_q^{2n+1})$ fixed by a further \mathcal{L} -action of a classical Lie algebra $\mathfrak{u}(1)$.

As the two left actions commute, both $\mathcal{A}(S_q^{2n+1})$ and $\mathcal{A}(\mathbb{CP}_q^n)$ are themselves left $U_q(\mathfrak{su}(n+1))$ -module $*$ -algebras for the action ‘ \triangleright ’.

Let us give few more details for the $n = 1$ case needed later on, while we refer to the literature for the $n > 1$ case. The Hopf $*$ -algebra $U_q(\mathfrak{su}(2))$ is generated by $K = K^*, K^{-1}, E, F = E^*$ with relations

$$KEK^{-1} = qE , \quad [E, F] = (K^2 - K^{-2})/(q - q^{-1}) ,$$

and coproduct/counit/antipode defined by

$$\Delta K = K \otimes K , \quad \Delta E = E \otimes K + K^{-1} \otimes E ,$$

$$\epsilon(K) = 1 , \quad \epsilon(E) = 0 ,$$

$$S(K) = K^{-1} , \quad S(E) = -qE .$$

One passes to the notations of [24] with the change $e = -F, f = -E, k = K$. The right canonical action is given, on generators α, β of $\mathcal{A}(SU_q(2))$, by

$$\begin{aligned} \alpha \triangleleft K &= q^{\frac{1}{2}} \alpha , & \alpha \triangleleft E &= -q\beta^* , & \alpha \triangleleft F &= 0 , \\ \beta \triangleleft K &= q^{\frac{1}{2}} \beta , & \beta \triangleleft E &= \alpha^* , & \beta \triangleleft F &= 0 . \end{aligned}$$

We finally need recalling that the representation theory of $U_q(\mathfrak{su}(2))$ is well known (cf. [40, Thm. 13]). In particular, we are interested in the irreducible representations in which K has positive spectrum: these are labelled by an integer $n \in \mathbb{N}$ with the representation space V_n of dimension $n+1$. In each of these the Casimir element of $U_q(\mathfrak{su}(2))$,

$$\mathcal{C}_q = \left(\frac{q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}}{q - q^{-1}} \right)^2 + FE, \quad (2)$$

has value $\mathcal{C}_q|_{V_n} = [\frac{n+1}{2}]^2 \text{id}_{V_n}$.

3 K-theory and K-homology

3.1 Equivariant modules and representations

Similarly to the construction of equivariant vector bundles associated to a principal bundle on a manifold, here we construct modules — that we interpret as sections of virtual ‘noncommutative equivariant vector bundles’ — as follows. Let $\sigma : U_q(\mathfrak{u}(n)) \rightarrow \text{End}(\mathbb{C}^k)$ be a $*$ -representation. The analogue of (sections of) the vector bundle associated to σ is the $\mathcal{A}(\mathbb{CP}_q^n)$ -module $\mathcal{E}(\sigma)$ of elements of $\mathcal{A}(SU_q(n+1)) \otimes \mathbb{C}^k$ which are $U_q(\mathfrak{u}(n))$ -invariant for the Hopf tensor product of the actions \mathcal{L} and σ . That is, $\psi \in \mathcal{A}(SU_q(n+1)) \otimes \mathbb{C}^k$ belongs to $\mathcal{E}(\sigma)$ if and only if

$$(\mathcal{L}_{x_{(1)}} \otimes \sigma(x_{(2)}))\psi = \epsilon(x)\psi, \quad \forall x \in U_q(\mathfrak{u}(n)). \quad (3)$$

As this set is stable under (left and right) multiplication by an $U_q(\mathfrak{u}(n))$ -element of $\mathcal{A}(SU_q(n+1))$, one has that $\mathcal{E}(\sigma)$ is an $\mathcal{A}(\mathbb{CP}_q^n)$ -bimodule. It is a left $\mathcal{A}(\mathbb{CP}_q^n) \rtimes U_q(\mathfrak{su}(n+1))$ -module as well, due to the ‘ \triangleright ’ and \mathcal{L} actions commuting.

Of particular importance are ‘line bundles’, — bimodules $\mathcal{E}(\sigma)$ coming from one-dimensional representations of $U_q(\mathfrak{u}(n)) \simeq U_q(\mathfrak{su}(n)) \oplus U(\mathfrak{u}(1))$, non-trivial only on the $\mathfrak{u}(1)$. Since the fixed point algebra $\mathcal{A}(SU_q(n+1))^{U_q(\mathfrak{su}(n))}$ coincides with $\mathcal{A}(S_q^{2n+1})$, (section of) noncommutative line bundles can be equivalently described as associated to the noncommutative $U(1)$ -principal bundle $S_q^{2n+1} \rightarrow \mathbb{CP}_q^n$ via an irreducible representation of $U(1)$. These are labelled by $N \in \mathbb{Z}$, and the general line bundle, that we denote by Γ_N , is given in §4 of [19]. They are all finitely generated and projective (as one-sided modules), as we shall explain in detail in §3.2. Note that $\Gamma_0 = \mathcal{A}(\mathbb{CP}_q^n)$.

The expressions are particularly simple for $n = 1$. In this case

$$\Gamma_N = \{a \in \mathcal{A}(SU_q(2)) \mid \mathcal{L}_K(a) = q^{\frac{N}{2}}a\}. \quad (4)$$

As a left $U_q(\mathfrak{su}(2))$ -module, we have a decomposition (cf. §2.2 of [55], where Γ_N is denoted M_{-N}):

$$\Gamma_N \simeq \bigoplus_{n-|N| \in 2\mathbb{N}} V_n, \quad (5)$$

where V_n is the irreducible representation of dimension $n+1$ mentioned before.

An $\mathcal{A}(\mathbb{CP}_q^n)$ -valued Hermitian structure on $\mathcal{E}(\sigma)$ is obtained by restriction of the canonical Hermitian structure of $\mathcal{A}(SU_q(n+1)) \otimes \mathbb{C}^k$, that is

$$(\psi, \eta)_{\mathcal{E}(\sigma)} := \sum_{i=1}^k \psi_i^* \eta_i,$$

for all $\psi = (\psi_1, \dots, \psi_k)$ and $\eta = (\eta_1, \dots, \eta_k)$, with $\psi_i, \eta_i \in \mathcal{A}(SU_q(n+1))$.

If instead of ‘representative functions’ $\mathcal{A}(SU_q(n+1))$ one works with the associated universal C^* -algebra $C(SU_q(n+1))$ of ‘continuous functions’, the above construction yields a full right Hilbert module over the C^* -algebra $C(\mathbb{CP}_q^n)$. Note that left multiplication by $a \in C(\mathbb{CP}_q^n)$ satisfy

$$(a\psi, a\psi)_{\mathcal{E}(\sigma)} = \sum_{i=1}^k \psi_i^* a^* a \psi_i \leq \|a\|^2 \sum_{i=1}^k \psi_i^* \psi_i = \|a\|^2 (\psi, \psi)_{\mathcal{E}(\sigma)}, \quad (6)$$

since $a^* a \leq \|a\|^2$ and conjugation with elements of a C^* -algebra preserves the positivity of an operator. Thus $\mathcal{E}(\sigma)$ is a Morita equivalence bimodule between $C(\mathbb{CP}_q^n)$ and $\text{End}_{C(\mathbb{CP}_q^n)} \mathcal{E}(\sigma)$ (cf. [42, App. A.3 and A.4]).

By composing the Hermitian structure with the Haar state one gets a pre-Hilbert space with inner product

$$\langle \psi, \psi' \rangle := h \circ (\psi, \psi')_{\mathcal{E}(\sigma)}. \quad (7)$$

From (6), it follows $\langle a\psi, a\psi' \rangle \leq \|a\|^2 \langle \psi, \psi' \rangle$, so that one has a bounded representation of $C(\mathbb{CP}_q^n)$ on the Hilbert space completion of each of these equivariant modules. These are the representations used in §4 for the construction of covariant differential calculi and equivariant spectral triples on \mathbb{CP}_q^n .

3.2 K-theory

At the C^* -algebra level, by viewing $C(\mathbb{CP}_q^n)$ as the Cuntz–Krieger algebra of a graph [36] one proves that $K_0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1}$ (and $K_1(C(\mathbb{CP}_q^n)) = 0$). The group K_0 is given as the cokernel of the incidence matrix canonically associated with the graph. The dual result for K-homology is obtained in an analogous way with the group K^0 being the kernel of the transposed matrix [14]; this leads to $K^0(C(\mathbb{CP}_q^n)) = \mathbb{Z}^{n+1}$ (and $K^1(C(\mathbb{CP}_q^n)) = 0$).

Somewhat implicitly, in [36] there appear generators of the K_0 groups of $C(\mathbb{CP}_q^n)$ as projections in $C(\mathbb{CP}_q^n)$ itself. In [21] we gave generators of

$K_0(C(\mathbb{CP}_q^n))$ in the form of ‘polynomial functions’, so they represent elements of $K_0(\mathcal{A}(\mathbb{CP}_q^n))$ as well. These latter generators are constructed as follows.

With $N \in \mathbb{Z}$, denote by $\Psi_N = (\psi_{j_0, \dots, j_n}^N)$ the vector-valued ‘function’ on S_q^{2n+1} with $\binom{|N|+n}{n}$ components given by:

$$\psi_{j_0, \dots, j_n}^N := [j_0, \dots, j_n]!^{\frac{1}{2}} q^{-\frac{1}{2} \sum_{r < s} j_r j_s} (z_0^*)^{j_0} \dots (z_n^*)^{j_n}, \quad \text{if } N \geq 0, \quad (8a)$$

$$\psi_{j_0, \dots, j_n}^N := [j_0, \dots, j_n]!^{\frac{1}{2}} q^{\frac{1}{2} \sum_{r < s} j_r j_s + \sum_{r=0}^n r j_r} z_0^{j_0} \dots z_n^{j_n}, \quad \text{if } N < 0, \quad (8b)$$

and labeled by non-negative integers satisfying $j_0 + \dots + j_n = |N|$. Then $\Psi_N^\dagger \Psi_N = 1$ and

$$P_N := \Psi_N \Psi_N^\dagger \quad (9)$$

is a projection: $(P_N)^2 = P_N = (P_N)^\dagger$; the proof is in [19, 21], and is a generalization of the case $n = 2$ in [22]. In particular $P_1 = P$ is the ‘defining’ projection of the algebra $\mathcal{A}(\mathbb{CP}_q^n)$ of §2.1. As we will see in §3.2, the group K_0 is generated by the classes of $P_0, P_{-1}, \dots, P_{-n}$.

The components of Ψ_N are a generating family for Γ_N as a left module, as shown in §4.1 of [19]; hence Γ_N is finitely generated and projective as a left module, and the corresponding projection is P_N . Also it is not difficult to prove that a generating family for Γ_N as a right module, is given by the components of Ψ_{-N}^\dagger ; hence Γ_N is finitely generated and projective as a right module too, with corresponding projection P_{-N} . For $n = 2$, this is Prop. 3.3 of [22] (what we call here Γ_N , following [19], is denoted $\Sigma_{0, -N}$ in [22]).

The projections P_N are ‘equivariant’ in the following sense. For an homogeneous space, the equivariant K^0 -group can be defined as the Grothendieck group of the abelian monoid whose elements are equivalence classes of equivariant vector bundles. It has an algebraic version, denoted $K_0^{\mathcal{U}}(\mathcal{A})$ where \mathcal{U} is a Hopf $*$ -algebra and \mathcal{A} a \mathcal{U} -module $*$ -algebra, valid in the non-commutative case as well. Equivariant vector bundles are replaced by one sided (left, say) $\mathcal{A} \rtimes \mathcal{U}$ -modules which are finitely generated and projective as (left) \mathcal{A} -modules; these will be simply called “equivariant projective modules”. Any such a module is given by a pair (p, σ) , where p is a $k \times k$ idempotent with entries in \mathcal{A} , and $\sigma : \mathcal{U} \rightarrow \text{Mat}_k(\mathbb{C})$ is a representation with the following compatibility requirement satisfied (see e.g. [17, Sect. 2]):

$$(x_{(1)} \triangleright p) \sigma(x_{(2)})^t = \sigma(x)^t p, \quad \text{for all } x \in \mathcal{U}, \quad (10)$$

with ‘ t ’ denoting transposition. The corresponding module $\mathcal{E} = \mathcal{A}^k p$ is made of row vectors elements $v = (v_1, \dots, v_k) \in \mathcal{A}^k$ in the range of the idempotent, $vp = v$, with module structures

$$(a.v)_i := av_i, \quad (x.v)_i := \sum_{j=1}^k (x_{(1)} \triangleright v_j) \sigma_{ij}(x_{(2)}), \quad i = 1, \dots, k,$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{U}$. An equivalence between two equivariant modules is simply an invertible left $\mathcal{A} \rtimes \mathcal{U}$ -module map between them. The group $K_0^{\mathcal{U}}(\mathcal{A})$ is defined as the Grothendieck group of the abelian monoid whose elements are equivalence classes of \mathcal{U} -equivariant projective \mathcal{A} -modules; the monoid operation is the direct sum, as usual.

There is an isomorphism $\Gamma_N \simeq \mathcal{A}(\mathbb{CP}_q^n)^{k_{N,n}} P_N$, with $k_{N,n} = \binom{|N|+n}{n}$, so that P_N are candidates to represent elements in equivariant K-theory. In fact, it is more convenient to take idempotents $P'_N = R_N P_N R_N^{-1}$, that are conjugated to P_N through the diagonal matrix R_N having component

$$q^{\frac{1}{2} \sum_{i=1}^n i(n+1-i)(j_{i-1}-j_i)} = q^{\frac{1}{2} \sum_{i=0}^n (n-2i)j_i}$$

in position (j_0, \dots, j_N) . The need to use idempotents that are not self-adjoint is explained in Lemma 2.7 of [17]: the module map $\Gamma_N \rightarrow \mathcal{A}(\mathbb{CP}_q^n)^{k_{N,n}} P_N$ is not unitary, while the map $\Gamma_N \rightarrow \mathcal{A}(\mathbb{CP}_q^n)^{k_{N,n}} P'_N$ is. For $n = 1$, these are exactly the projections in [55, Eq. (33)].

Proposition 3.1 *The pair (P'_N, σ^N) is the representative of an element in $K_0^{U_q(\mathfrak{su}(n+1))}(\mathcal{A}(\mathbb{CP}_q^n))$. Here, σ^N is the irreducible representation with highest weight $(N, 0, \dots, 0)$ if $N \geq 0$, or with highest weight $(0, \dots, 0, -N)$ if $N < 0$.*

Proof. Let us give the proof for $N < 0$, the one for $N \geq 0$ being similar. We write the components of Ψ_N as ψ_J^N , where $J = (j_0, \dots, j_n)$ is a multi-index.

The explicit formulæ for the action on z_j 's are in §4 of [19]. One has

$$E_i \triangleright z'_j = \delta_{i,j} z'_{j+1}, \quad F_i \triangleright z'_j = \delta_{i,j+1} z'_i, \quad K_i \triangleright z'_j = q^{\frac{1}{2}(\delta_{i+1,j} - \delta_{i,j})} z'_j, \quad (11)$$

where $\{E_i, F_i, K_i\}$ are the generators of $U_q(\mathfrak{su}(n+1))$ and z'_j of $\mathcal{A}(S_q^{2n+1})$ in the notations of [19], with $i = 1, \dots, n$, and $j = 1, \dots, n+1$; we recall that our present notations differ from the ones in [19] for a replacement $z_i = z'_{n+1-i}$.

For a fixed N , let V_N be the linear span of the components ψ_J^N of Ψ_N . Then V_N carry a representation of $U_q(\mathfrak{su}(n+1))$. And $\psi_{-N,0,\dots,0}^N = (z_0)^{-N}$ is the highest weight vector of the representation $(0, \dots, 0, -N)$, as $K_i \triangleright (z_0)^{-N} = 0$ for all $i \neq n$ and $K_n \triangleright (z_0)^{-N} = q^{-\frac{1}{2}N} (z_0)^{-N}$, and $E_i \triangleright (z_0)^{-N} = 0$ for all i . Hence the representation $\tilde{\sigma}^N$ on V_N defined by

$$x \triangleright \psi_J^N = \sum_{J'} \psi_{J'}^N \tilde{\sigma}_{J',J}^N(x) \quad (12)$$

contains the irreducible representation $(0, \dots, 0, -N)$. Having the latter dimension $k_{N,n} = \binom{|N|+n}{n}$ by Weyl character formula (cf. Lemma 3.4 of [19]), and being this the dimension of V_N , the two representations coincide. Let

$$\sigma^N(x) := R_N \tilde{\sigma}^N(x) R_N^{-1}.$$

In matrix notations, (12) becomes $x \triangleright R_N \Psi_N = \sigma^N(x)^t R_N \Psi_N$, thinking of Ψ_N as a column vector and with row-by-column multiplication understood. Also,

$$\begin{aligned} (x \triangleright \Psi_N^\dagger) R_N^{-1} &= (S(x)^* \triangleright \Psi_N)^\dagger R_N^{-1} = (\tilde{\sigma}^N(S(x)^*)^t \Psi_N)^\dagger R_N^{-1} \\ &= \Psi_N^\dagger \tilde{\sigma}^N(S(x))^t R_N^{-1} = \Psi_N^\dagger R_N \sigma^N(S(x))^t R_N^{-2}. \end{aligned}$$

Therefore

$$\begin{aligned} (x_{(1)} \triangleright P'_N) \sigma^N(x_{(2)})^t &= (x_{(1)} \triangleright R_N \Psi_N) (x_{(2)} \triangleright \Psi_N^\dagger R_N^{-1}) \sigma^N(x_{(3)})^t \\ &= \sigma^N(x_{(1)})^t P'_N R_N^2 \sigma^N(S(x_{(2)}))^t R_N^{-2} \sigma^N(x_{(3)})^t. \end{aligned}$$

We need (also for later use in §7.4), the element $K_{2\rho}$ — implementing the square of the antipode — and given in [19, eq. (3.2)]:

$$K_{2\rho} = \left(K_1^n K_2^{2(n-1)} \dots K_j^{j(n-j+1)} \dots K_n^n \right)^2. \quad (13)$$

For now, one readily checks that $K_{2\rho} \triangleright \Psi_N = q^{\sum_{i=1}^n i(n+1-i)(j_{i-1}-j_i)} \Psi_N$, so that $R_N = \sigma^N(K_{2\rho}^{\frac{1}{2}})^t$ and

$$R_N^2 \sigma^N(S(x))^t R_N^{-2} = \sigma^N(K_{2\rho}^{-1} S(x) K_{2\rho}) = \sigma^N(S^{-1}(x))^t \quad (14)$$

for all $x \in U_q(\mathfrak{su}(n+1))$, by (3.3) of [19]. Thus,

$$\begin{aligned} (x_{(1)} \triangleright P'_N) \sigma^N(x_{(2)})^t &= \sigma^N(x_{(1)})^t P'_N \sigma^N(S^{-1}(x_{(2)}))^t \sigma^N(x_{(3)})^t \\ &= \sigma^N(x_{(1)})^t P'_N \sigma^N(x_{(3)} S^{-1}(x_{(2)}))^t \\ &= \sigma^N(x_{(1)})^t P'_N \epsilon(x_{(2)}) = \sigma^N(x)^t P'_N, \end{aligned}$$

that is exactly (10). ■

3.3 Fredholm modules and Chern characters

As we already mentioned, for \mathbb{CP}_q^n the group K^1 is trivial. Here we describe the group K^0 , whose elements are represented by Fredholm modules over $\mathcal{A}(\mathbb{CP}_q^n)$. An even Fredholm module $(\mathcal{A}, \mathcal{H}, F, \gamma)$ over a $*$ -algebra \mathcal{A} is a \mathbb{Z}_2 -graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (with grading operator γ), together with

a graded representation $\pi = \pi_+ \oplus \pi_-$ of \mathcal{A} on \mathcal{H} and an odd bounded self-adjoint operator F such that $F^2 = 1$ and $[F, \pi(a)]$ is a compact operator for all $a \in \mathcal{A}$. If $[F, \pi(a)]$ is of trace class for all $a \in \mathcal{A}$, we say that the Fredholm module is 1-summable. The representation symbol will be usually omitted.

Among the generators of $K^0(\mathcal{A}(\mathbb{CP}_q^n))$ there is one whose representation is faithful, which we call ‘top’ Fredholm module and describe firstly.

Let $\underline{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ and $|\underline{m}\rangle$ be the canonical orthonormal basis of $\ell^2(\mathbb{N}^n)$. For $0 \leq i < k \leq n$, we denote by $\underline{\varepsilon}_i^k \in \{0, 1\}^n$ the array

$$\underline{\varepsilon}_i^k := (\overbrace{0, 0, \dots, 0}^{i \text{ times}}, \overbrace{1, 1, \dots, 1}^{k-i \text{ times}}, \overbrace{0, 0, \dots, 0}^{n-k \text{ times}}).$$

Definition 3.2 ([21]) Let $0 \leq k \leq n$ and $\mathcal{V}_k^n \subset \ell^2(\mathbb{N}^n)$ be the linear span of basis vectors $|\underline{m}\rangle$ satisfying the constraints $0 \leq m_1 \leq m_2 \leq \dots \leq m_k$ and $m_{k+1} > m_{k+2} > \dots > m_n \geq 0$, with $m_0 := 0$. For any $k > 0$, a representation $\pi_{n,k} : \mathcal{A}(S_q^{2n+1}) \rightarrow \mathcal{B}(\ell^2(\mathbb{N}^n))$ is defined as follows. We set $\pi_{n,k}(z_i) = 0$ for all $i > k \geq 1$, while the remaining generators are

$$\begin{aligned} \pi_{n,k}(z_i) |\underline{m}\rangle &= q^{m_i} \sqrt{1 - q^{2(m_{i+1} - m_i + 1)}} |\underline{m} + \underline{\varepsilon}_i^k\rangle, \quad \text{for } 0 \leq i \leq k-1, \\ \pi_{n,k}(z_k) |\underline{m}\rangle &= q^{m_k} |\underline{m}\rangle, \end{aligned}$$

on the subspace $\mathcal{V}_k^n \subset \ell^2(\mathbb{N}^n)$, and they are zero on the orthogonal subspace. When $k = 0$, we define $\pi_{n,0}(z_i) = 0$ if $i > 0$, while

$$\begin{aligned} \pi_{n,0}(z_0) |\underline{m}\rangle &= |\underline{m}\rangle, & \text{for } m_1 > m_2 > \dots > m_n \geq 0, \\ \pi_{n,0}(z_0) |\underline{m}\rangle &= 0, & \text{otherwise.} \end{aligned}$$

Each representation $\pi_{n,k}$ is an irreducible $*$ -representation of both $\mathcal{A}(S_q^{2n+1})$ and $\mathcal{A}(\mathbb{CP}_q^n)$ when restricted to \mathcal{V}_k^n , and is identically zero outside \mathcal{V}_k^n . Most importantly, if $|j - k| > 1$ we have [21]:

$$\pi_{n,j}(a) \pi_{n,k}(b) = 0, \quad \forall a, b \in \mathcal{A}(S_q^{2n+1}).$$

As a consequence, the maps $\pi_{\pm} : \mathcal{A}(S_q^{2n+1}) \rightarrow \mathcal{B}(\ell^2(\mathbb{N}^n))$, defined by

$$\pi_+^{(n)}(a) := \sum_{\substack{k \text{ even} \\ 0 \leq k \leq n}} \pi_{n,k}(a), \quad \pi_-^{(n)}(a) := \sum_{\substack{k \text{ odd} \\ 0 \leq k \leq n}} \pi_{n,k}(a),$$

are representations of the algebra $\mathcal{A}(S_q^{2n+1})$ and, by restriction, of $\mathcal{A}(\mathbb{CP}_q^n)$.

An even Fredholm module for $\mathcal{A}(\mathbb{CP}_q^n)$ is obtained with the representation $\pi_n := \pi_+^{(n)} \oplus \pi_-^{(n)}$ on $\mathcal{H}_n := \ell^2(\mathbb{N}^n) \oplus \ell^2(\mathbb{N}^n)$, obvious grading operator γ_n , and

$$F_n := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its 1-summability follows from the proposition below [21].

Proposition 3.3 *The difference $\pi_+^{(n)}(a) - \pi_-^{(n)}(a)$ is of trace class on \mathcal{H}_n for all $a \in \mathcal{A}(\mathbb{CP}_q^n)$. Furthermore, the trace is given by a series which — as a function of q — is absolutely convergent in the open interval $0 < q < 1$.*

Additional n Fredholm modules $(\mathcal{A}(\mathbb{CP}_q^n), \mathcal{H}_k, F_k, \gamma_k)$, $0 \leq k < n$, are obtained using the $*$ -algebra morphism $\mathcal{A}(\mathbb{CP}_q^n) \rightarrow \mathcal{A}(\mathbb{CP}_q^k)$, restriction of the morphism $\mathcal{A}(S_q^{2n+1}) \rightarrow \mathcal{A}(S_q^{2k+1})$ given by the map sending to zero the generators $z_{k+1}, z_{k+2}, \dots, z_n$. With this map, one pull-backs the ‘top’ Fredholm module of \mathbb{CP}_q^n to \mathbb{CP}_q^k . For $k = 0$, we set $\mathcal{A}(\mathbb{CP}_q^0) := \mathbb{C}$ and the ‘top’ Fredholm module — the generator of $K^0(\mathbb{C})$ —, is given by the (non-unital) representation $\mathbb{C} \ni a \mapsto a \oplus 0$ on $\mathcal{H}_0 := \mathbb{C} \oplus \mathbb{C}$, with grading $\gamma_0 = 1 \oplus -1$ and F_0 the operator interchanging the two components, $F_0(x \oplus y) = y \oplus x$ for all $x, y \in \mathbb{C}$.

The pairing of the K-homology class $[F_k]$ of $(\mathcal{A}(\mathbb{CP}_q^n), \mathcal{H}_k, F_k, \gamma_k)$ with an element $[p] \in K_0(\mathcal{A}(\mathbb{CP}_q^n))$ is given by:

$$\langle [F_k], [p] \rangle = \frac{1}{2} \operatorname{Tr}(\gamma_k F_k [F_k, p]) .$$

In particular, for $k = 0$ this computes the dimension of the fiber of the restriction of noncommutative vector bundles over \mathbb{CP}_q^n to the ‘classical point’ of \mathbb{CP}_q^n (given by the unique character of the algebra): the computation yields the ‘rank’ of the corresponding projective module.

Proposition 3.4 *For any $N \in \mathbb{N}$ and for all $0 \leq k \leq n$, the pairing between the K-theory classes $[P_{-N}]$ of the (line bundle) projections P_{-N} described in §3.2 and the K-homology classes $[F_k]$ is:*

$$\langle [F_k], [P_{-N}] \rangle = \binom{N}{k} ,$$

with $\binom{N}{k} := 0$ when $k > N$.

For the proof in [21] one computes the pairing by evaluating the series giving the trace in the $q \rightarrow 0$ limit, being the series absolutely convergent as in Prop. 3.3. For $q \rightarrow 0$ only finitely many terms survive, and the final result easily follows. With the above result, we proved in [21] that the elements $[F_0], \dots, [F_n]$ are generators of $K^0(\mathcal{A}(\mathbb{CP}_q^n))$, and the elements $[P_0], \dots, [P_{-n}]$ are generators of $K_0(\mathcal{A}(\mathbb{CP}_q^n))$. In particular, similarly to the classical situation, the K-theory is generated by line bundles.

4 Dirac operators and spectral triples

4.1 Regular spectral triples

Spectral triples, or “unbounded Fredholm modules”, provide a non-commutative generalization of the notion of closed Riemannian orientable (or spin^c) manifold [7, 9]. A unital spectral triple is the datum $(\mathcal{A}, \mathcal{H}, D)$ of a $*$ -algebra \mathcal{A} with a bounded representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ on a Hilbert space \mathcal{H} , and a selfadjoint operator D on \mathcal{H} — the ‘Dirac’ operator — with compact resolvent, such that $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}$. The spectral triple is called *even* if $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is \mathbb{Z}_2 -graded and, for this decomposition, $\pi(\mathcal{A})$ is diagonal while the operator D is off-diagonal. We denote by γ the grading operator, and set $\gamma = 1$ when the spectral triple is *odd* (no grading then). The compact resolvent requirement for the Dirac operator guarantees, for example, in the even case that the twisting of $D_\pm = D|_{\mathcal{H}_\pm}$ with projections are Fredholm operators: a crucial property for the construction of ‘topological invariants’ via index computations [7]. If there is a $d \in \mathbb{R}^+$ such that $(1 + D^2)^{-d/2}$ is in the Dixmier ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, the spectral triple is said to have “metric dimension” d or to be d -summable (cf. Chap. 4 of [7]).

While spectral triples correspond to spin^c or orientable Riemannian manifolds, *real* spectral triples correspond to manifolds that are spin [8]. A spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is *real* if there is in addition an antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$, called the *real structure*, such that

$$J^2 = \epsilon 1, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J, \quad (15)$$

and

$$[a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0, \quad (16)$$

for all $a, b \in \mathcal{A}$. The signs ϵ , ϵ' and ϵ'' determine the KO-dimension (an integer modulo 8) of the triple [8]. In some examples (not in the present case) conditions (16) have to be slightly relaxed (see for instance [24]).

As conformal structures are classes of (pseudo-)Riemannian metrics, similarly Fredholm modules are “conformal classes” of spectral triples [7, 2]: given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, a Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ can be obtained by replacing D with the bounded operator $F := D(1 + D^2)^{-\frac{1}{2}}$ (one can use $F := D|D|^{-1}$ if D is invertible), and viceversa any K-homology class has a representative that arises from a spectral triple through this construction [1]. Passing from bounded to unbounded Fredholm modules is convenient since it allows to use powerful tools such as local index formulæ [11].

From now on we shall only consider spectral triples whose representation is faithful, identify \mathcal{A} with $\pi(\mathcal{A})$ and omit the representation symbol. For the space \mathbb{CP}_q^n , we introduced in [21] even spectral triples of any metric dimension $d \in \mathbb{R}^+$ whose conformal class is the top Fredholm module $(\mathcal{A}(\mathbb{CP}_q^n), \mathcal{H}_n, F_n, \gamma_n)$ in §3.3. They are constructed by giving explicitly the spectral decomposition of the Dirac operator. For example, an n -dimensional spectral triple is obtained by taking $D = |D|F_n$ on \mathcal{H}_n with

$$|D| |\underline{m}\rangle := (m_1 + \dots + m_n) |\underline{m}\rangle .$$

The eigenvalues are $\pm\lambda$, for $\lambda \in \mathbb{N}$, with multiplicity $\binom{\lambda+n}{n-1}$. This is a polynomial in λ of order $n-1$, and so the metric dimension is n , as claimed.

4.2 Equivariant spectral triples

When \mathcal{A} is a \mathcal{U} -module $*$ -algebra, for some Hopf $*$ -algebra \mathcal{U} , one may consider spectral triples with “symmetries”, describing the analogue of homogeneous spin structures. A unital spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is called \mathcal{U} -equivariant if

- (i) there is a dense subspace $\mathcal{M} \subset \text{Dom}(D)$ of \mathcal{H} where the representation of \mathcal{A} can be extended to a representation of $\mathcal{A} \rtimes \mathcal{U}$,
- (ii) both D and γ commute with \mathcal{U} on \mathcal{M} .

In case there is a real structure J , one further asks that $J|_{\mathcal{M}}$ is the antiunitary part of a (possibly unbounded) antilinear operator $\tilde{J} : \mathcal{M} \rightarrow \mathcal{M}$ such that

$$\tilde{J}x = S(x)^* \tilde{J}, \quad \forall x \in \mathcal{U}. \quad (17)$$

In other words, the antilinear involutive automorphism $x \mapsto S(x)^*$ of the Hopf $*$ -algebra \mathcal{U} is implemented by the operator \tilde{J} . This resonates with a known feature of quantum-group duality in the C^* -algebra setting of [47], where, in that setting, it is discussed the relation of the Tomita operator with the antipode S and the $*$ structure of a quantum group.

As already mentioned, spectral triples for \mathbb{CP}_q^n were constructed in [21]. They are typical of the noncommutative case, as they have no $q \rightarrow 1$ analogue. Although they are not equivariant, they are “regular” in the sense of [11].

On the other hand, even regular spectral triples on q -spaces usually don't give very interesting local index formulæ; typically the unique term surviving in Connes-Moscovici local cocycle [11] is the non-local one (cf. [16, 17]). On \mathbb{CP}_q^1 a more complicated local index formula is given in [50], and is obtained using the *non-regular* and *equivariant* spectral triple of [25]. The geometrical

nature of the latter is explained in [52] where it is also implicitly suggested how to generalize the construction to \mathbb{CP}_q^n , by using the action of the Hopf algebra $U_q(\mathfrak{su}(n+1))$, in particular the action of quasi-primitive elements, which are external derivations on \mathbb{CP}_q^n . This idea was used in [41] to construct — on any quantum irreducible flag manifold, including then \mathbb{CP}_q^n — a Dirac operator D that realizes by commutators the unique covariant (irreducible, finite-dimensional) first order $*$ -calculus of [34] for \mathbb{CP}_q^n . In particular, the exterior derivative is given by $\delta(a) = \sqrt{-1} [D, a]$, for $a \in \mathbb{CP}_q^n$ (the coefficient $\sqrt{-1}$ is inserted to get a real derivation). It is not clear whether this leads to a spectral triples or not, as the compact resolvent condition is yet unproven.

Equivariant spectral triples on \mathbb{CP}_q^n are constructed in [19], in complete analogy with the $q = 1$ case, by using the fact that complex projective spaces are Kähler manifolds: in particular they admit a homogeneous (for the action of $SU(n+1)$) Kähler metric, the *Fubini-Study metric*. The result is a family of (equivariant, even) spectral triples $(\mathcal{A}(\mathbb{CP}_q^n), \mathcal{H}_N, D_N, \gamma_N)$ labelled by $N \in \mathbb{Z}$ (Although we use the same symbol, the Hilbert spaces here are not the Hilbert spaces \mathcal{H}_k of §3.). The space \mathcal{H}_0 are the noncommutative analogue of $(0, 1)$ -forms (in fact, they give a finite-dimensional covariant differential calculus on \mathbb{CP}_q^n) with D_0 the analogue of the Dolbeault-Dirac operator; \mathcal{H}_N is the tensor product of \mathcal{H}_0 with ‘sections of line bundles’ with monopole charge N over \mathbb{CP}_q^n with D_N the twisting of D_0 by the Grassmannian connection of the line bundle. If n is odd, for $N = \frac{1}{2}(n+1)$ one has a *real* spectral triple whose Dirac operator is a deformation of the Dirac operator of the Fubini-Study metric, in parallel with \mathbb{CP}^n being a spin manifold when n is odd.

The spectrum of D_N is computed by relating its square D_N^2 to the Casimir of $U_q(\mathfrak{su}(n+1))$. One finds that for $q < 1$ the eigenvalues of D_N grow exponentially, hence the spectral triple is of metric dimension 0^+ , or better 0^+ -summable. For $q = 1$ one finds (as expected) the spectrum given in [26].

5 The projective line \mathbb{CP}_q^1 as a noncommutative manifold

Recall the $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules Γ_N in (4). As proven in [25], modulo unitary equivalences there is a unique real equivariant even spectral triple for \mathbb{CP}_q^1 on the Hilbert space completion of $\Gamma_1 \oplus \Gamma_{-1}$. This spectral triple has metric dimension 0 (there is no real equivariant spectral triple on $\Gamma_1 \oplus \Gamma_{-1}$ with summability different from 0^+). Its geometrical nature — that the Dirac operator is coming from the right action of $U_q(\mathfrak{su}(2))$ —, is explained in [52].

Twisting the Dirac operator on the tensor product of $\Gamma_1 \oplus \Gamma_{-1}$ with a line bundle [53], leads to spectral triples which are in general not real [19, Sect. 2]. In [16] we constructed spectral triples of any summability with a real structure J satisfying a weaker version of the reality and first order condition in (16).

In the next section we shall describe a new family of equivariant real spectral triples for \mathbb{CP}_q^1 . They generalize the ones of [25] and are all inequivalent to each other (in particular, not equivalent to the one of [25]).

5.1 A family of equivariant real spectral triples for \mathbb{CP}_q^1

With the $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules Γ_N in (4), for $n \in \frac{1}{2}\mathbb{Z}$, let W_n be the Hilbert space completion of Γ_{-2n} with respect to the inner product coming from the Haar state of $SU_q(2)$ as in (7). For a fixed $j \in \mathbb{N} + \frac{1}{2}$, we call \mathcal{H}_j the space of vectors $\mathbf{a} = (a_{-j}, a_{-j+1}, \dots, a_j)^t$ with components $a_n \in W_n$, for $n = -j, -j+1, \dots, j$.

The representation of $\mathcal{A}(\mathbb{CP}_q^1)$ is the obvious left module structure of \mathcal{H}_j . The Dirac operator D_j , the grading γ_j and the real structure J_j are given by

$$\begin{aligned} D_j \mathbf{a} &= (\mathcal{L}_E a_{-j+1}, \mathcal{L}_F a_{-j}, \mathcal{L}_E a_{-j+3}, \mathcal{L}_F a_{-j+2}, \dots, \mathcal{L}_E a_j, \mathcal{L}_F a_{j-1})^t, \\ \gamma_j \mathbf{a} &= (-a_{-j}, a_{-j+1}, -a_{-j+2}, a_{-j+3}, \dots, -a_{j-1}, a_j)^t, \\ J_j \mathbf{a} &= K \triangleright (q^{-j} a_j^*, -q^{-j+1} a_{j-1}^*, \dots, q^{j-1} a_{-j+1}^*, -q^j a_{-j}^*)^t. \end{aligned}$$

Note that $\gamma_j|_{W_n}$ is 1 if $j+n$ is odd and is -1 if $j+n$ is even.

Proposition 5.1 *The datum $(\mathcal{A}(\mathbb{CP}_q^1), \mathcal{H}_j, D_j, \gamma_j, J_j)$ is a real even $U_q(\mathfrak{su}(2))$ -equivariant spectral triple, with KO -dimension 2 and metric dimension 0^+ .*

Proof. The proof is analogous to the one in [52] (and generalizations in [18, 19]). By definition $a_n \in \Gamma_{-2n}$ satisfies $\mathcal{L}_K(a_n) = q^{-n} a_n$, and $\mathcal{L}_K \mathcal{L}_E = q \mathcal{L}_E \mathcal{L}_K$ proves that \mathcal{L}_E is a densely defined operator $W_n \rightarrow W_{n-1}$. Similarly, \mathcal{L}_F is a densely defined operator $W_n \rightarrow W_{n+1}$. Hence D_j is a well defined symmetric operator on $\mathcal{M} := \bigoplus_n \Gamma_{2n}$. It can be closed to a self-adjoint operator on \mathcal{H}_j (in fact, one can diagonalize it and give its domain of self-adjointness explicitly).

The representation of $\mathcal{A}(\mathbb{CP}_q^1)$ is clearly bounded. From the coproduct formula of E , and the defining property $\mathcal{L}_K(a) = a$ of $a \in \mathcal{A}(\mathbb{CP}_q^1)$, one gets

$$\mathcal{L}_E(a\eta) = (\mathcal{L}_E a)(\mathcal{L}_{K^{-1}}\eta) + (\mathcal{L}_K a)(\mathcal{L}_E \eta) = q^n (\mathcal{L}_E a)\eta + a(\mathcal{L}_E \eta),$$

for all $a \in \mathcal{A}(\mathbb{CP}_q^1)$ and $\eta \in \Gamma_{-2n}$. Thus the commutator $[\mathcal{L}_E, a] = q^n \mathcal{L}_E(a)$, is the multiplication operator for an element $\mathcal{L}_E(a) \in \mathcal{A}(SU_q(2))$, and hence

a bounded operator $W_n \rightarrow W_{n-1}$. A similar formula holds for $[\mathcal{L}_F, a]$, proving that $[D_j, a]$ is a bounded operator, for all $a \in \mathcal{A}(\mathbb{CP}_q^1)$.

The grading commutes with any $a \in \mathcal{A}(\mathbb{CP}_q^1)$ and anticommutes with D_j . The square of the Dirac operator is

$$D_j^2 \mathbf{a} = (\mathcal{L}_E \mathcal{L}_F a_{-j}, \mathcal{L}_F \mathcal{L}_E a_{-j+1}, \dots, \mathcal{L}_E \mathcal{L}_F a_j, \mathcal{L}_F \mathcal{L}_E a_{j-1})^t.$$

Since \mathcal{L}_K is proportional to the identity on each Γ_{-2n} , modulo a constant matrix, D_j^2 is given by the \mathcal{L} action of the central element \mathcal{C}_q in (2). For a central element left and right canonical actions coincide, and with respect to the left action of $U_q(\mathfrak{su}(2))$ we have the decomposition into irreducible representations $\Gamma_{-2n} \simeq \bigoplus_{\ell=|n| \in \mathbb{N}} V_{2\ell}$, as given in (5). Since the eigenvalues of \mathcal{C}_q grow exponentially with ℓ , the operator \mathcal{C}_q has compact resolvent on each W_n ; the operator D_j has compact resolvent too, given that there are only finitely many W_n in \mathcal{H}_j . This proves that $(\mathcal{A}(\mathbb{CP}_q^1), \mathcal{H}_j, D_j, \gamma_j)$ is a spectral triple. In fact, the eigenvalues of D_j growing exponentially as well, the operator $(1 + D_j^2)^{-\epsilon}$ is of trace class for any $\epsilon > 0$. Hence the metric dimension of the spectral triple is 0^+ (the spectral triple is 0^+ -summable).

Next the real structure. The operator J_j is an isometry:

$$\begin{aligned} \langle J_j \mathbf{a}, J_j \mathbf{b} \rangle &= \sum_n q^{-2n} h((K^{-1} \triangleright a_n)(K \triangleright b_n^*)) \\ &= \sum_n h((K^{-1} \triangleright a_n \triangleleft K^{-1})(K \triangleright b_n^* \triangleleft K)) \\ &= \sum_n h((K \triangleright b_n^* \triangleleft K)(K \triangleright a_n \triangleleft K)) \\ &= \sum_n h(K \triangleright (b_n^* a_n) \triangleleft K) \\ &= \sum_n h(b_n^* a_n) = \langle \mathbf{b}, \mathbf{a} \rangle, \end{aligned}$$

where we used the bi-invariance and the modular property of the Haar state h , i.e. $h(ab) = h(b(K^2 \triangleright a \triangleleft K^2))$ (cf. eq. (3.4) in [19]). Clearly $J_j \gamma_j = -\gamma_j J_j$. And, since $x \triangleright a^* = (S(x)^* \triangleright a)^*$ for all $x \in U_q(\mathfrak{su}(2))$ and any $a \in \mathcal{A}(SU_q(2))$, one also easily checks that $J_j^2 = -1$. As for the antilinear operator \tilde{J} as in (17), let $\tilde{J}_j : \mathcal{M} \rightarrow \mathcal{M}$ be the (unbounded) operator

$$\tilde{J}_j \mathbf{a} = (q^{-j} a_j^*, -q^{-j+1} a_{j-1}^*, \dots, q^{j-1} a_{-j+1}^*, -q^j a_{-j}^*)^t.$$

Note that $J_j \mathbf{a} = K \triangleright (\tilde{J}_j \mathbf{a})$. Since $K \triangleright$ is a positive operator, J_j is the antiunitary part of \tilde{J}_j . Furthermore, from $x \triangleright a^* = (S(x)^* \triangleright a)^*$ and $S(S(x)^*) = x$ it follows $S(x^*) \triangleright \tilde{J}_j(\mathbf{a}) = \tilde{J}_j(x \triangleright \mathbf{a})$ for all $x \in U_q(\mathfrak{su}(2))$, i.e. the relation (17).

Since $\mathcal{L}_E(a^*) = -q^{-1}(\mathcal{L}_F a)^*$ and $\mathcal{L}_F(a^*) = -q(\mathcal{L}_E a)^*$, we have:

$$\begin{aligned}
D_j \tilde{J}_j \mathbf{a} &= \left(-q^{-j+1} \mathcal{L}_E(a_{j-1}^*), q^{-j} \mathcal{L}_F(a_j^*), \dots, -q^j \mathcal{L}_E(a_{-j}^*), q^{j-1} \mathcal{L}_F(a_{-j+1}^*) \right)^t \\
&= \left(q^{-j} (\mathcal{L}_F a_{j-1})^*, -q^{-j+1} (\mathcal{L}_E a_j)^*, \dots, q^{j-1} (\mathcal{L}_F a_{-j})^*, -q^j (\mathcal{L}_E a_{-j+1})^* \right)^t \\
&= \tilde{J}_j (\mathcal{L}_E a_{-j+1}, \mathcal{L}_F a_{-j}, \mathcal{L}_E a_{-j+3}, \mathcal{L}_F a_{-j+2}, \dots, \mathcal{L}_E a_j, \mathcal{L}_F a_{j-1})^t \\
&= \tilde{J}_j D_j \mathbf{a}.
\end{aligned}$$

As left and right actions of $U_q(\mathfrak{su}(2))$ commute, it follows that $[D_j, J_j] = 0$.

The signs in (15) are then $\epsilon = -1$, $\epsilon' = 1$ and $\epsilon'' = -1$ and correspond to KO-dimension 2. One easily checks that $J_j a J_j^{-1}$ is the operator of right multiplication by a^* , for all $a \in \mathcal{A}(\mathbb{CP}_q^1)$, and hence it commutes with b and $[D_j, b]$ for $b \in \mathcal{A}(\mathbb{CP}_q^1)$. This proves both conditions (16).

The left action of $U_q(\mathfrak{su}(2))$ on \mathcal{M} commutes with D_j (it commutes with the right action), it clearly commutes with the grading, since each Γ_{-2n} is a left $U_q(\mathfrak{su}(2))$ -module, and in fact the representation of $\mathcal{A}(\mathbb{CP}_q^1)$ extends to a representation of $\mathcal{A}(\mathbb{CP}_q^1) \rtimes U_q(\mathfrak{su}(2))$. Thus we have a real equivariant spectral triple, as claimed. \blacksquare

Spectral triples of Prop. 5.1 corresponding to different values of j are ‘topologically’ inequivalent since, as we shall see in next section, they give different values when paired with the generator $p = P_1$ of $K_0(\mathcal{A}(\mathbb{CP}_q^1))$,

$$p = \begin{pmatrix} \alpha^* \alpha & \alpha^* \beta \\ \beta^* \alpha & \beta^* \beta \end{pmatrix} = \begin{pmatrix} 1 - q^2 A & B^* \\ B & A \end{pmatrix}, \quad (18)$$

having used generators $A = \beta^* \beta$ and $B = \beta^* \alpha$ for the algebra $\mathcal{A}(\mathbb{CP}_q^1)$.

In the next section this result will be also used to establish rational Poincaré duality for the spectral triples, thus generalizing the analogous result proven in [55] for the spectral triple of [25].

5.2 Index computations and rational Poincaré duality

We consider here the spectral triple $(\mathcal{A}(\mathbb{CP}_q^1), \mathcal{H}_j, D_j, \gamma_j, J_j)$ of Prop. 5.1, being $j \in \mathbb{N} + \frac{1}{2}$ a fixed number. With $n \in \mathbb{Z} + \frac{1}{2}$ and condition $|n| \leq j$, let

$$\mathcal{H}_j^+ := (1 + \gamma_j) \mathcal{H}_j = \bigoplus_{j+n \text{ odd}} W_n, \quad \mathcal{H}_j^- := (1 - \gamma_j) \mathcal{H}_j = \bigoplus_{j+n \text{ even}} W_n,$$

and let $D_j^+ := D_j|_{\mathcal{H}_j^+} \otimes \text{id}_{\mathbb{C}^2}$. Let p be the ‘defining’ projection in (18). We aim at computing the index of the (unbounded) operator

$$pD_j^+ p : p(\mathcal{H}_j^+ \otimes \mathbb{C}^2) \rightarrow p(\mathcal{H}_j^- \otimes \mathbb{C}^2) ,$$

yielding the pairing of the K-homology class of the spectral triple with the non-trivial generator of $K_0(\mathcal{A}(\mathbb{CP}_q^1))$.

Proposition 5.2 *It holds that*

$$\text{Index}(pD_j^+ p) = \begin{cases} \frac{1}{2}(j^2 - \frac{9}{4}) & \text{if } j \in 2\mathbb{N} + \frac{1}{2} , \\ \frac{1}{2}(j^2 - \frac{1}{4}) & \text{if } j \in 2\mathbb{N} + \frac{3}{2} . \end{cases}$$

The index being never zero, these spectral triples are “topologically” non-trivial.

Proof. To compute the index, we look for a “nice” basis of $\mathcal{H}_j^\pm \otimes \mathbb{C}^2$. We begin by recalling the left regular representation of $\mathcal{A}(SU_q(2))$, found for instance in [24]. An orthonormal basis of $\mathcal{A}(SU_q(2))$ is given by

$$|l, m, n\rangle = q^n [2l+1]^{\frac{1}{2}} t_{nm}^l , \quad l \in \frac{1}{2}\mathbb{N}, \quad l - |m| \in \mathbb{N}, \quad l - |n| \in \mathbb{N}.$$

with t_{nm}^l the matrix elements of irreducible corepresentations [40, Sect. 4.2.4] (with respect to the notations of [24] we exchanged the labels m and n). The left regular representation is given on generators by [24, Prop. 3.3]

$$\begin{aligned} \alpha |l, m, n\rangle &= q^{-l+\frac{1}{2}(m+n-1)} \left(\frac{[l+m+1][l+n+1]}{[2l+1][2l+2]} \right)^{\frac{1}{2}} |l^+, m^+, n^+\rangle \\ &\quad + q^{l+\frac{1}{2}(m+n+1)} \left(\frac{[l-m][l-n]}{[2l][2l+1]} \right)^{\frac{1}{2}} |l^-, m^+, n^+\rangle , \\ \beta |l, m, n\rangle &= q^{\frac{1}{2}(m+n-1)} \left(\frac{[l-m+1][l+n+1]}{[2l+1][2l+2]} \right)^{\frac{1}{2}} |l^+, m^-, n^+\rangle \\ &\quad - q^{\frac{1}{2}(m+n-1)} \left(\frac{[l+m][l-n]}{[2l][2l+1]} \right)^{\frac{1}{2}} |l^-, m^-, n^+\rangle , \end{aligned}$$

with the notation $k^\pm := k \pm \frac{1}{2}$. Also, from [24, eq. (3.1)] and the definition of the automorphism ϑ there (i.e. $K = k = \vartheta(k^{-1})$, $E = -f = \vartheta(e)$ and $F = -e = \vartheta(f)$) we deduce

$$\begin{aligned} \mathcal{L}_K |l, m, n\rangle &= q^{-n} |l, m, n\rangle , \\ \mathcal{L}_F |l, m, n\rangle &= \sqrt{[l-n][l+n+1]} |l, m, n+1\rangle , \\ \mathcal{L}_E |l, m, n\rangle &= \sqrt{[l-n+1][l+n]} |l, m, n-1\rangle . \end{aligned}$$

The Hilbert space W_n has basis $|l, m, n\rangle$, with $n \in \mathbb{Z} + \frac{1}{2}$ fixed, $l = |n|, |n|+1, \dots$ and $m = -l, -l+1, \dots, l$. Using this, a basis of $W_n \otimes \mathbb{C}^2$ (already employed in [15, Sect. 3.8]), is given by:

$$v_{l,m}^{n,\uparrow} := \frac{1}{\sqrt{[2l]}} \begin{pmatrix} \sqrt{q^{-l+m}[l+m]} |l^-, m^-, n\rangle \\ \sqrt{q^{l+m}[l-m]} |l^-, m^+, n\rangle \end{pmatrix}, \quad l = |n| + \frac{1}{2}, |n| + \frac{3}{2}, \dots,$$

$$v_{l,m}^{n,\downarrow} := \frac{1}{\sqrt{[2l+2]}} \begin{pmatrix} \sqrt{q^{l+m+1}[l-m+1]} |l^+, m^-, n\rangle \\ -\sqrt{q^{-l+m-1}[l+m+1]} |l^+, m^+, n\rangle \end{pmatrix},$$

$$l = |n| - \frac{1}{2}, |n| + \frac{1}{2}, \dots,$$

where $m = -l, -l+1, \dots, l$. Notice that in previous equation l and m are integers (while n is not). For notational convenience, we set $v_{|n|-\frac{1}{2},m}^{n,\uparrow} := 0$ and start counting from $l = |n| - \frac{1}{2}$ for both v^\uparrow and v^\downarrow . An easy exercise checks that passing from the vectors $\{|l, m, n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |l, m, n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ to the vectors $\{v_{l,m}^{n,\uparrow}, v_{l,m}^{n,\downarrow}\}$ is an isometry, and thus we got an orthonormal basis of $W_n \otimes \mathbb{C}^2$.

The restriction of the left regular representation of $\mathcal{A}(SU_q(2))$ to $\mathcal{A}(\mathbb{CP}_q^1)$ is given on generators A and B by:

$$\begin{aligned} A |l, m, n\rangle &= -q^{m+n-1} \frac{1}{[2l+2]} \sqrt{\frac{[l+m+1][l-m+1][l+n+1][l-n+1]}{[2l+1][2l+3]}} |l+1, m, n\rangle \\ &\quad + q^{m+n-1} \left(\frac{[l-m+1][l+n+1]}{[2l+1][2l+2]} + \frac{[l+m][l-n]}{[2l][2l+1]} \right) |l, m, n\rangle \\ &\quad - q^{m+n-1} \frac{1}{[2l]} \sqrt{\frac{[l+m][l-m][l+n][l-n]}{[2l-1][2l+1]}} |l-1, m, n\rangle, \\ B |l, m, n\rangle &= -q^{-l+m+n-\frac{1}{2}} \frac{1}{[2l+2]} \sqrt{\frac{[l+m+1][l+m+2][l+n+1][l-n+1]}{[2l+1][2l+3]}} |l+1, m+1, n\rangle \\ &\quad + q^{m+n} \frac{\sqrt{[l+m+1][l-m]}}{[2l+1]} \left(\frac{q^{-l-\frac{1}{2}}[l+n+1]}{[2l+2]} - \frac{q^{l+\frac{1}{2}}[l-n]}{[2l]} \right) |l, m+1, n\rangle \\ &\quad + q^{l+m+n+\frac{1}{2}} \frac{1}{[2l]} \sqrt{\frac{[l-m][l-m-1][l+n][l-n]}{[2l-1][2l+1]}} |l-1, m+1, n\rangle. \end{aligned}$$

Using these, for the action of the projection p in (18), we get:

$$\begin{aligned} p v_{l,m}^{n,\uparrow} &= q^{-l+n-\frac{1}{2}} \frac{[l+n+\frac{1}{2}]}{[2l+1]} v_{l,m}^{n,\uparrow} + q^n \frac{\sqrt{[l+n+\frac{1}{2}][l-n+\frac{1}{2}]}}{[2l+1]} v_{l,m}^{n,\downarrow}, \\ p v_{l,m}^{n,\downarrow} &= q^n \frac{\sqrt{[l+n+\frac{1}{2}][l-n+\frac{1}{2}]}}{[2l+1]} v_{l,m}^{n,\uparrow} + q^{l+n+\frac{1}{2}} \frac{[l-n+\frac{1}{2}]}{[2l+1]} v_{l,m}^{n,\downarrow}, \end{aligned}$$

if $l > |n| - \frac{1}{2}$, while for $l = |n| - \frac{1}{2}$:

$$p v_{|n|-\frac{1}{2},m}^{n,\downarrow} = \begin{cases} 0 & \text{if } n > 0, \\ v_{|n|-\frac{1}{2},m}^{n,\downarrow} & \text{if } n < 0. \end{cases}$$

We rewrite previous equations in the form

$$\begin{pmatrix} p v_{l,m}^{n,\uparrow} \\ p v_{l,m}^{n,\downarrow} \end{pmatrix} = \frac{q^n}{[2l+1]} \times$$

$$\begin{aligned}
 & \times \begin{pmatrix} q^{-l-\frac{1}{2}}[l+n+\frac{1}{2}] & \sqrt{[l+n+\frac{1}{2}][l-n+\frac{1}{2}]} \\ \sqrt{[l+n+\frac{1}{2}][l-n+\frac{1}{2}]} & q^{l+\frac{1}{2}}[l-n+\frac{1}{2}] \end{pmatrix} \begin{pmatrix} v_{l,m}^{n,\uparrow} \\ v_{l,m}^{n,\downarrow} \end{pmatrix} \\
 & =: \begin{pmatrix} P_{l,m,n}^{11} & P_{l,m,n}^{12} \\ P_{l,m,n}^{12} & P_{l,m,n}^{22} \end{pmatrix} \begin{pmatrix} v_{l,m}^{n,\uparrow} \\ v_{l,m}^{n,\downarrow} \end{pmatrix}
 \end{aligned}$$

and notice that — for any l, m, n — the 2×2 matrix

$$P_{l,m,n} = \begin{pmatrix} P_{l,m,n}^{11} & P_{l,m,n}^{12} \\ P_{l,m,n}^{12} & P_{l,m,n}^{22} \end{pmatrix}$$

is a rank 1 projection. Thus, since $(P_{l,m,n}^{11})^2 + (P_{l,m,n}^{12})^2 = P_{l,m,n}^{11}$, the matrix

$$R_{l,m,n} := \frac{1}{\sqrt{P_{l,m,n}^{11}}} \begin{pmatrix} P_{l,m,n}^{11} & P_{l,m,n}^{12} \\ P_{l,m,n}^{12} & -P_{l,m,n}^{11} \end{pmatrix},$$

is a rotation. It is, in fact, unipotent, i.e. $R_{l,m,n}^2 = 1$. The vectors

$$\begin{pmatrix} w_{l,m}^{n,||} \\ w_{l,m}^{n,\perp} \end{pmatrix} := R_{l,m,n} \begin{pmatrix} v_{l,m}^{n,\uparrow} \\ v_{l,m}^{n,\downarrow} \end{pmatrix}$$

together with $v_{|n|-\frac{1}{2},m}^{n,\downarrow}$ form an orthonormal basis of $W_n \otimes \mathbb{C}^2$ made of eigenvectors of the projection p , i.e.

$$p w_{l,m}^{n,||} = w_{l,m}^{n,||}, \quad p w_{l,m}^{n,\perp} = 0.$$

The space $p(\mathcal{H}_N^+ \otimes \mathbb{C}^2)$ is the span of the vectors:

$$\begin{aligned}
 w_{l,m}^{n,||} & \quad \forall \quad n = -j+1, -j+3, \dots, j, \quad l = |n| + \frac{1}{2}, |n| + \frac{3}{2}, \dots, \\
 v_{|n|-\frac{1}{2},m}^{n,\downarrow} & \quad \forall \quad n = -j+1, -j+3, \dots, j : n < 0,
 \end{aligned}$$

and for all $m = -l, -l+1, \dots, l$, with $l = |n| - \frac{1}{2}$ in the latter case.

Similarly $p(\mathcal{H}_N^- \otimes \mathbb{C}^2)$ is the span of the vectors:

$$\begin{aligned}
 w_{l,m}^{n,||} & \quad \forall \quad n = -j, -j+2, \dots, j-1, \quad l = |n| + \frac{1}{2}, |n| + \frac{3}{2}, \dots, \\
 v_{|n|-\frac{1}{2},m}^{n,\downarrow} & \quad \forall \quad n = -j, -j+2, \dots, j-1 : n < 0,
 \end{aligned}$$

and for all $m = -l, -l+1, \dots, l$, with $l = |n| - \frac{1}{2}$ in the latter case.

On the vectors $\{w_{l,m}^{n,||}, w_{l,m}^{n,\perp}\}$, the action of \mathcal{L}_E and \mathcal{L}_F will have the form

$$\begin{pmatrix} \mathcal{L}_E w_{l,m}^{n,||} \\ \mathcal{L}_E w_{l,m}^{n,\perp} \end{pmatrix} = \begin{pmatrix} A_{l,m,n} & \dots \\ \dots & \dots \end{pmatrix} \begin{pmatrix} w_{l,m}^{n-1,||} \\ w_{l,m}^{n-1,\perp} \end{pmatrix},$$

and

$$\begin{pmatrix} \mathcal{L}_F w_{l,m}^{n,||} \\ \mathcal{L}_F w_{l,m}^{n,\perp} \end{pmatrix} = \begin{pmatrix} B_{l,m,n} & \dots \\ \dots & \dots \end{pmatrix} \begin{pmatrix} w_{l,m}^{n+1,||} \\ w_{l,m}^{n+1,\perp} \end{pmatrix}.$$

A vector $w_{l,m}^{n,||}$ is in the kernel of pD_j^+ if and only if $A_{l,m,n} = 0$, and is in the cokernel if and only if $B_{l,m,n} = 0$. Using the action of \mathcal{L}_E and \mathcal{L}_F , found out to be given by

$$\begin{aligned} \mathcal{L}_E v_{l,m}^{n,\uparrow} &= \sqrt{[l-n+\frac{1}{2}][l+n-\frac{1}{2}]} v_{l,m}^{n-1,\uparrow}, \\ \mathcal{L}_E v_{l,m}^{n,\downarrow} &= \sqrt{[l-n+\frac{3}{2}][l+n+\frac{1}{2}]} v_{l,m}^{n-1,\downarrow}, \\ \mathcal{L}_F v_{l,m}^{n,\uparrow} &= \sqrt{[l-n-\frac{1}{2}][l+n+\frac{1}{2}]} v_{l,m}^{n+1,\uparrow}, \\ \mathcal{L}_F v_{l,m}^{n,\downarrow} &= \sqrt{[l-n+\frac{1}{2}][l+n+\frac{3}{2}]} v_{l,m}^{n+1,\downarrow}, \end{aligned}$$

a straightforward computation shows that:

$$\begin{aligned} B_{l,m,n} = A_{l,m,n+1} &= \sqrt{[l-n-\frac{1}{2}][l+n+\frac{1}{2}]} \frac{P_{l,m,n}^{11} P_{l,m,n+1}^{11}}{\sqrt{P_{l,m,n}^{11} P_{l,m,n+1}^{11}}} \\ &+ \sqrt{[l-n+\frac{1}{2}][l+n+\frac{3}{2}]} \frac{P_{l,m,n}^{12} P_{l,m,n+1}^{12}}{\sqrt{P_{l,m,n}^{11} P_{l,m,n+1}^{11}}}. \end{aligned}$$

Thus if the vector $w_{l,m}^{n,||}$ is in the kernel of pD_j^+ , the vector $w_{l,m}^{n-1,||}$ is in the cokernel, so that they give no contribution to the index of $pD_j^+ p$, and the index depends only on the vectors $v_{|n|-\frac{1}{2},m}^{n,\downarrow}$. From the action above, one finds that for any $n < 0$,

$$\mathcal{L}_E v_{|n|-\frac{1}{2},m}^{n,\downarrow} = 0, \quad \mathcal{L}_F v_{|n|-\frac{1}{2},m}^{n,\downarrow} = \sqrt{[-2n]} v_{|n|-\frac{1}{2},m}^{n+1,\downarrow} \neq 0.$$

The vector $v_{|n|-\frac{1}{2},m}^{n+1,\downarrow}$ is in the image of p if $n+1 < 0$ and is in the kernel if $n+1 > 0$. Thus, all $v_{|n|-\frac{1}{2},m}^{n,\downarrow}$ belonging to $p(\mathcal{H}_j^+ \otimes \mathbb{C}^2)$ are in the kernel of $pD_j^+ p$ while in the cokernel we have only $v_{0,0}^{-1/2,\downarrow}$, and only in the case it belongs to $p(\mathcal{H}_j^- \otimes \mathbb{C}^2)$, i.e. when $j \in 2\mathbb{N} + \frac{1}{2}$. We distinguish then three cases:

1. if $j = \frac{1}{2}$, then $\text{Index}(pD_j^+ p) = -1$;
2. if $j = 2k + \frac{3}{2} \in 2\mathbb{N} + \frac{3}{2}$, then

$$\begin{aligned}
 \text{Index}(pD_j^+ p) &= \sum_{n=-j+1, -j+3, \dots, -1/2} (-2n) \\
 &= \sum_{i=0}^k (4i+1) = (2k+1)(k+1) = \frac{1}{2}(j^2 - \frac{1}{4}) ;
 \end{aligned}$$

3. if $j = 2k + \frac{5}{2} \in 2\mathbb{N} + \frac{5}{2}$, then

$$\begin{aligned}
 \text{Index}(pD_j^+ p) &= \sum_{n=-j+1, -j+3, \dots, -3/2} (-2n) - 1 = \sum_{i=0}^k (4i+3) - 1 \\
 &= (2k+1)(k+2) = \frac{1}{2}(j^2 - \frac{9}{4}) .
 \end{aligned}$$

Note that the equation at point 3. gives the correct answer also for $j = \frac{1}{2}$. Since for $k \in \mathbb{N}$, $(2k+1)(k+1)$ and $(2k+1)(k+2)$ are strictly positive, the index is never zero. \blacksquare

As anticipated then:

Corollary 5.3 *Since for different values of j we get different values of $\text{Index}(pD_j^+ p)$, the spectral triples $(\mathcal{A}(\mathbb{CP}_q^1), \mathcal{H}_j, D_j, \gamma_j, J_j)$ in Prop. 5.1 correspond to distinct K -homology classes.*

Rational Poincaré duality allows one to prove several interesting estimates on the eigenvalues of the twist of the Dirac operator D with a Hermitian finitely generated projective modules, (cf. [48, Thm. 1]). Let us recall its definition [7] for the particular case of a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ with $K_1(\mathcal{A}) = 0$. One says that the spectral triple satisfies rational Poincaré duality if the pairing $\langle \cdot, \cdot \rangle_D : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \rightarrow \mathbb{Z}$, defined by

$$\langle [P], [Q] \rangle_D := \text{Index}(P \otimes JQJ^*)D_j^+(P \otimes JQJ^*) ,$$

is non-degenerate, for P a $r \times r$ and Q a $s \times s$ projection. Here, $P \otimes JQJ^*$ is a projection on $\mathcal{H} \otimes \mathbb{C}^{rs}$ and $D_j^+ = D_j|_{\mathcal{H}_+} \otimes \text{id}_{\mathbb{C}^{rs}}$.

The generators of $K_0(\mathcal{A}(\mathbb{CP}_q^1)) \simeq \mathbb{Z}^2$ can be taken to be the class of the trivial projector [1] corresponding to $(1, 0)$ and the class of the projector p in (18) corresponding [46, 32] to $(1, 1)$. Thus a generic element of $K_0(\mathcal{A}(\mathbb{CP}_q^1))$ can be labelled, with $i, k \in \mathbb{Z}$, as

$$(i, k) = (i - k) [1] + k[p] .$$

Proposition 5.4 *The spectral triples $(\mathcal{A}(\mathbb{CP}_q^1), \mathcal{H}_j, D_j, \gamma_j, J_j)$ of Prop. 5.1 satisfy rational Poincaré duality, for any $j \in \mathbb{N} + \frac{1}{2}$. In particular, the pairing is given by the explicit formula:*

$$\langle (i, k), (i', k') \rangle_{D_j} = (ki' - ik') \langle [p], [1] \rangle_{D_j}, \quad (19)$$

where $\langle [p], [1] \rangle_{D_j} = \text{Index}(pD_j^+ p)$ is the index computed in Prop. 5.2.

Proof. One repeats the first part of the proof of [55, Prop. 7.4] showing the antisymmetry of the pairing induced by the Dirac operator; hence by bilinearity it is always of the form (19), where $\langle [p], [1] \rangle_{D_j} = \text{Index}(pD_j^+ p)$ is the index computed in Prop. 5.2 and is different from zero for all values of j . Since

$$\langle (i, k), (k, -i) \rangle_{D_j} = (i^2 + k^2) \langle [p], [1] \rangle_{D_j}$$

is equal to zero only if $i = k = 0$, the pairing is non-degenerate: for any not zero element (i, k) there is at least another not zero element $(k, -i)$ such that the pairing of the two is not zero. This concludes the proof. \blacksquare

6 A digression: calculi and connections

6.1 Covariant differential calculi

A differential $*$ -calculus over a $*$ -algebra \mathcal{A} is a differential graded $*$ -algebra $(\Omega^\bullet(\mathcal{A}), d)$ with $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^{k+1}(\mathcal{A}) = \text{Span}\{a d\omega, a \in \mathcal{A}, \omega \in \Omega^k\}$, for all $k \geq 0$. Requesting a graded Leibniz rule, the differential is uniquely determined by its restriction to 0-forms. The datum (\mathcal{M}, δ) of an \mathcal{A} -bimodule and a real derivation $\delta : \mathcal{A} \rightarrow \mathcal{M}$, i.e. such that $\delta(a^*) = \delta(a)^*$, is called a *first order* $*$ -calculus; generality is not lost by assuming that $\mathcal{M} = \text{Span}\{a db, a, b \in \mathcal{A}\}$, as if this is not the case one can replace \mathcal{M} with the obvious sub-bimodule. A canonical way to construct a differential $*$ -calculus from (\mathcal{M}, δ) is to define $\Omega^0(\mathcal{A}) = \mathcal{A}$, $\Omega^1(\mathcal{A}) = \mathcal{M}$ and $\Omega^{k+1}(\mathcal{A}) = \Omega^k(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, with product given by the tensor product over \mathcal{A} ; the derivation δ uniquely extends to a differential d giving a differential $*$ -calculus on \mathcal{A} . On the other hand, one can take any other graded $*$ -algebra $\Omega^\bullet(\mathcal{A})$ having $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^1(\mathcal{A}) = \mathcal{M}$, and (uniquely) extend the derivation using the graded Leibniz rule.

A derivation $\delta_u : \mathcal{A} \rightarrow \ker m$, with image the kernel of the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is given by $\delta_u a := a \otimes 1 - 1 \otimes a$, $a \in \mathcal{A}$, and for any other derivation $\delta : \mathcal{A} \rightarrow \mathcal{M}$ there exists a bimodule map $j : \ker m \rightarrow \mathcal{M}$ such that $\delta = j \circ \delta_u$; in this sense δ_u is universal (see e.g. [13]).

If \mathcal{U} is a Hopf $*$ -algebra, one says that the calculus $(\Omega^\bullet(\mathcal{A}), d)$ is \mathcal{U} -covariant if $\Omega^\bullet(\mathcal{A})$ is a graded left \mathcal{U} -module $*$ -algebra (i.e. the action of \mathcal{U} is a degree zero map, thus respecting the grading) and d commutes with the action of \mathcal{U} . It follows that $\Omega^k(\mathcal{A})$ is a left $\mathcal{A} \rtimes \mathcal{U}$ -module for any $k \geq 1$.

6.2 Complex structures

Suppose $(\Omega^\bullet(M), d)$ is the de Rham complex of a smooth manifold M . Note that what here we denote by Ω^k are *complex valued* k -forms. From an algebraic point of view, an almost complex structure is a decomposition $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ of 1-forms into a $(1, 0)$ and a $(0, 1)$ part, and this induces a corresponding decomposition $\Omega^k(M) = \bigoplus_{r+s=k} \Omega^{r,s}(M)$. The wedge product of forms is a bi-graded product, i.e. $\Omega^{p,q}(M) \wedge \Omega^{r,s}(M) \subset \Omega^{p+r, q+s}(M)$, and the involution sends $\Omega^{r,s}(M)$ into $\Omega^{s,r}(M)$. Denoting by $\pi_{r,s}$ the projection $\Omega^{r+s}(M) \rightarrow \Omega^{r,s}(M)$, one can decompose the differential as

$$d|_{\Omega^{p,q}(M)} = \sum_{r+s=p+q+1} \pi_{r,s} \circ d|_{\Omega^{p,q}(M)} = \partial + \bar{\partial} + \dots,$$

where $\partial = \pi_{p+1,q} \circ d|_{\Omega^{p,q}}$ has degree $(1, 0)$ and $\bar{\partial} = \pi_{p,q+1} \circ d|_{\Omega^{p,q}}$ has degree $(0, 1)$. If M is a *complex manifold*, then $d = \partial + \bar{\partial}$ without the additional terms (in general one may have terms of degree $(2, -1)$, $(-1, 2)$, etc.).

From $d = \partial + \bar{\partial}$ and $d^2 = 0$ it follows that $\partial^2 = 0$, $\bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$ (since d^2 is the sum of the three maps ∂^2 , $\bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial$, and they have different degree). In fact, an almost complex manifold is a complex manifold when one of the following equivalent conditions is satisfied (§1.3 of [57]):

- the Lie bracket of $(1, 0)$ vector fields is of type $(1, 0)$ (dually to the decomposition of 1-forms one has the analogous decomposition of vector fields);
- $d = \partial + \bar{\partial}$;
- $\bar{\partial}^2 = 0$.

The second one is what we use to define complex noncommutative spaces.

Definition 6.1 *A complex structure on an algebra \mathcal{A} equipped with a differential $*$ -calculus $(\Omega^\bullet(\mathcal{A}), d)$ is a bi-graded $*$ -algebra $\Omega^{\bullet,\bullet}(\mathcal{A})$ with two linear maps $\partial : \Omega^{\bullet,\bullet}(\mathcal{A}) \rightarrow \Omega^{\bullet+1,\bullet}(\mathcal{A})$ and $\bar{\partial} : \Omega^{\bullet,\bullet}(\mathcal{A}) \rightarrow \Omega^{\bullet,\bullet+1}(\mathcal{A})$ such that $\Omega^k(\mathcal{A}) = \bigoplus_{p+q=k} \Omega^{p,q}(\mathcal{A})$ and $d = \partial + \bar{\partial}$.*

The corresponding Dolbeault complex is the differential complex

$$\mathcal{A} \xrightarrow{\bar{\partial}} \Omega^{0,1}(\mathcal{A}) \xrightarrow{\bar{\partial}} \Omega^{0,2}(\mathcal{A}) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{0,n}(\mathcal{A}) \xrightarrow{\bar{\partial}} \dots \quad (20)$$

Note that the condition that d is a graded derivation is equivalent to both ∂ and $\bar{\partial}$ be graded derivations while $d(a^*) = d(a)^*$ is equivalent to $\bar{\partial}a = \partial(a^*)^*$.

The algebra of “holomorphic elements”, $\mathcal{O}(\mathcal{A}) := \ker \{\bar{\partial} : \mathcal{A} \rightarrow \Omega^{(0,1)}(\mathcal{A})\}$, is indeed an algebra over \mathbb{C} by the Leibniz rule. Its elements will be referred to, if a bit loosely, as holomorphic functions.

6.3 Connections

Let $(\Omega^\bullet(\mathcal{A}), d)$ be a differential calculus over a $*$ -algebra \mathcal{A} and \mathcal{E} a right \mathcal{A} -module. An (*affine*) *connection* on \mathcal{E} is a \mathbb{C} -linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ satisfying the Leibniz rule,

$$\nabla(\eta a) = \nabla(\eta)a + \eta \otimes_{\mathcal{A}} da, \quad \forall \eta \in \mathcal{E} \ a \in \mathcal{A}, \quad (21)$$

By the graded Leibniz rule, any connection is extended uniquely to a \mathbb{C} -linear map $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^\bullet(\mathcal{A}) \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{\bullet+1}(\mathcal{A})$. Due to the Leibniz rule, the *curvature* $\nabla^2 : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A})$ is right \mathcal{A} -linear, $\nabla^2(\eta a) = \nabla^2(\eta)a$, i.e. it is an element in $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}))$.

Connections for to the universal differential calculus are called themselves universal. Their importance is twofold: i) a universal connection on a module \mathcal{E} exists if and only if \mathcal{E} is projective [13]; ii) given a universal connection ∇_u on \mathcal{E} and a calculus $(\Omega^1(\mathcal{A}), d)$, called $j : \ker m \rightarrow \Omega^1(\mathcal{A})$ the bimodule map intertwining the differentials — i.e. such that $d = j \circ \delta_u$ —, one constructs a connection ∇ for the latter calculus using the formula $\nabla := (\text{id} \otimes j) \circ \nabla_u$. For $\mathcal{E} = p\mathcal{A}^k$ a finitely generated projective module, this connection (also named the *Grassmannian connection* of \mathcal{E}) is given by

$$\nabla_p \eta = p d\eta,$$

with d acting diagonally on \mathcal{A}^k , and row-by-column multiplication is understood. Being the space of all connection an affine space, any other connection differs from ∇_p by an element in $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}))$.

Affine connections on left modules are defined in a similar manner.

If \mathcal{E} is a bimodule, one defines a *bimodule connection* as a pair (∇, σ) of a right module connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ and a bimodule isomorphism $\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that $\sigma^{-1} \circ \nabla$ is a left module connection. Explicitly, this means (cf. [42, Sect. 8.5]) there is the *left* Leibniz rule as well:

$$\nabla(a\eta) = a\nabla(\eta) + \sigma(da \otimes_{\mathcal{A}} \eta), \quad \forall a \in \mathcal{A}, \eta \in \mathcal{E}.$$

Given bimodule connections (∇_i, σ_i) on \mathcal{A} -bimodules \mathcal{E}_i , $i = 1, 2$, a bimodule connection (∇, σ) on $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ can be defined (cf. [37, Prop. 2.12]) by $\nabla = (1 \otimes \sigma_2)(\nabla_1 \otimes 1) + 1 \otimes \nabla_2$ and $\sigma = (\text{id} \otimes \sigma_2)(\sigma_1 \otimes \text{id})$.

6.4 Holomorphic structures on modules

Given a complex structure on an algebra \mathcal{A} as in Definition 6.1, a *holomorphic connection* on a left \mathcal{A} -module \mathcal{E} is simply a connection

$$\nabla^{\bar{\partial}} : \mathcal{E} \rightarrow \Omega^{0,1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

for the differential calculus $(\Omega^{0,\bullet}(\mathcal{A}), \bar{\partial})$. The connection is called *integrable* or *flat* if its curvature vanishes: $(\nabla^{\bar{\partial}})^2 = 0$. In this case, the pair $(\mathcal{E}, \nabla^{\bar{\partial}})$ is a *holomorphic module* (cf. [37, Sect. 2]). Similar definitions are available for right modules.

For an integrable connection, in analogy with (20), one has a complex

$$\mathcal{E} \xrightarrow{\nabla^{\bar{\partial}}} \Omega^{0,1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\nabla^{\bar{\partial}}} \dots \xrightarrow{\nabla^{\bar{\partial}}} \Omega^{0,n}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\nabla^{\bar{\partial}}} \dots$$

The zero-th cohomology group of this complex, $H^0(\mathcal{E}, \nabla^{\bar{\partial}})$, will be called the “space of holomorphic sections” of \mathcal{E} . By the Leibniz rule it is, in fact, a (left) module over the algebra $\mathcal{O}(\mathcal{A})$ of holomorphic functions.

7 The complex structure of \mathbb{CP}_q^n

A first attempt to classify first order differential calculi on $\mathcal{A}(\mathbb{CP}_q^n)$ is in [56], where it is proven that, for $n \geq 5$, there exists a unique differential calculus if one requires some (pretty strong) constraints. One of these is the requirement (cf. [56, Sect. 4.2]) that $\Omega^1(\mathbb{CP}_q^n)$ is a free left module of rank $n(n+2)$, quite a strong one, given that the cotangent bundle of \mathbb{CP}^n is not parallelizable — the module of sections is not free —, and the rank is n (as a complex vector bundle). In that paper there is also a discussion of first order calculi on $\mathcal{A}(\mathbb{CP}_q^n)$ that are the restriction of calculi on $\mathcal{A}(S_q^{2n+1})$. Few years later, it was proven in [34] that for \mathbb{CP}_q^n there is only one covariant (irreducible, finite-dimensional) first order $*$ -calculus. Higher order differential calculi are studied in [35]. As already mentioned, this first order differential calculus can be realized by commutators with a “Dirac operator” [41]. The calculus was re-obtained in [3] as the restriction of a distinguished quotient of the bicovariant calculus on $\mathcal{A}(SU_q(n+1))$.

We then proceed to complex and related holomorphic structures on \mathbb{CP}_q^n . This started in [37] for \mathbb{CP}_q^1 , later generalized to \mathbb{CP}_q^2 in [38] and \mathbb{CP}_q^n in [39].

7.1 The Dolbeault complex

For \mathbb{CP}_q^n , the differential Dolbeault complex as in (20) has been constructed in [19]. Roughly speaking, forms $\Omega^{0,k}(\mathbb{CP}_q^n)$ are given by the equivariant module associated to the irreducible $*$ -representation of $U_q(\mathfrak{su}(n))$ with highest weight

$$\left(\overbrace{0, \dots, 0}^{k-1 \text{ times}}, 1, \overbrace{0, \dots, 0}^{n-k-1 \text{ times}} \right),$$

for any $1 \leq k \leq n-1$, extended, the representation, to $U_q(\mathfrak{u}(n))$ in a way that the element $\hat{K} := (K_1 K_2 \dots K_n)^{\frac{2}{n+1}}$ (cf. eq. (3.1) in [19]) is q^k times the identity. The module $\Omega^{0,n}(\mathbb{CP}_q^n)$ is simply Γ_{-n-1} . A $(0, k)$ -form is a vector $\omega = (\omega_{i_1, i_2, \dots, i_k})$ having components $\omega_{i_1, i_2, \dots, i_k} \in \mathcal{A}(SU_q(n+1))$, with labels satisfying the constraints $1 \leq i_1 < i_2 < \dots < i_k \leq n$, and transforming under the \mathcal{L} -action of $U_q(\mathfrak{u}(n))$ according to the above-mentioned representation. The product of forms is denoted by \wedge_q and given by

$$(\omega \wedge_q \omega')_{i_1, \dots, i_{h+k}} = \sum_{p \in S_{h+k}^{(h)}} (-q^{-1})^{|p|} \omega_{i_{p(1)}, \dots, i_{p(h)}} \omega'_{i_{p(h+1)}, \dots, i_{p(h+k)}}$$

for all $\omega \in \Omega^{0,h}(\mathbb{CP}_q^n)$ and $\omega' \in \Omega^{0,k}(\mathbb{CP}_q^n)$, and for all $h, k = 0, \dots, n$ with $h+k \leq n$. Moreover, we set $v \wedge_q \omega := 0$ if $h+k > n$. Here $S_{h+k}^{(h)}$ are permutations whose inverse is a (h, k) -shuffle, and $|p|$ is the length of the permutation p . The details in [19] show that the above is a well-defined associative product. Notice that, for any $a \in \mathcal{A}(\mathbb{CP}_q^2)$ it holds that

$$\omega \wedge_q a \omega' = \omega a \wedge_q \omega',$$

meaning that the product is a quotient of the free tensor product over $\mathcal{A}(\mathbb{CP}_q^2)$.

Any product of 1-forms, ω^i , $i = 1, \dots, k$, is also easy to describe:

$$(\omega^1 \wedge_q \omega^2 \wedge_q \dots \wedge_q \omega^k)_{i_1, \dots, i_k} = \sum_{p \in S_k} (-q^{-1})^{|p|} \omega_{i_{p(1)}}^1 \omega_{i_{p(2)}}^2 \dots \omega_{i_{p(k)}}^k,$$

with S_k the group of permutations of k objects. For $q = 1$ this is the antisymmetric tensor product over the algebra.

A graded derivation $\bar{\partial} : \Omega^{0,k}(\mathbb{CP}_q^n) \rightarrow \Omega^{0,k+1}(\mathbb{CP}_q^n)$, of the form

$$\bar{\partial} = \sum_1^n \mathcal{L}_{\hat{K} X_i}, \quad (22)$$

for suitable elements $X_i \in U_q(\mathfrak{u}(n+1))$ [19, eq. (5.7)], squares to zero, $(\bar{\partial})^2 = 0$, thus giving a covariant (Dolbeault-like) differential calculus $(\Omega^{0,\bullet}(\mathcal{A}(\mathbb{CP}_q^n)), \bar{\partial})$. Moreover, the map $h \rightarrow \mathcal{L}_h$ being a $*$ -representation, the Hermitian conjugate operator,

$$\bar{\partial}^\dagger = \sum_1^n \mathcal{L}_{X_i^* \hat{K}} ,$$

maps $\Omega^{0,k}(\mathbb{CP}_q^n)$ to $\Omega^{0,k-1}(\mathbb{CP}_q^n)$ and squares to zero as well: $(\bar{\partial}^\dagger)^2 = 0$. One needs stressing that the above calculus is not a $*$ -calculus. For later use, we mention that the element X_i above are given by [19, Lemma 3.13]:

$$X_i := N_{i,n} M_{i,n}^* \quad (23)$$

with $N_{i,n}$ the elements [19, eq. (3.7)]:

$$N_{i,n} := (K_i K_{i+1} \cdots K_n) \hat{K}^{-1} , \quad i = 1, \dots, n ,$$

whereas the elements $M_{i,n}$ are defined recursively [19, eq. (3.5)] by:

$$M_{i,n} = E_i M_{i+1,n} - q^{-1} M_{i+1,n} E_i , \quad i = 1, \dots, n .$$

7.2 Hodge decomposition and Dolbeault cohomology

We are ready to compute the cohomology groups $H_{\bar{\partial}}^\bullet(\mathbb{CP}_q^n)$ of the complex $(\Omega^{0,\bullet}(\mathcal{A}(\mathbb{CP}_q^n)), \bar{\partial})$ by generalizing the analogue of the Hodge decomposition theorem envisaged in [18] for the case $n = 2$. Let

$$\Delta_{\bar{\partial}} = (\bar{\partial} + \bar{\partial}^\dagger)^2$$

be the Hodge Laplacian. We call *harmonic* $(0, k)$ -forms the collection:

$$\mathfrak{H}^{0,k}(\mathbb{CP}_q^n) = \{ \omega \in \Omega^{0,k}(\mathbb{CP}_q^n) \mid \Delta_{\bar{\partial}} \omega = 0 \} .$$

Thus $\omega \in \Omega^{0,k}(\mathbb{CP}_q^n)$ is harmonic if and only if it is in the kernel of $\bar{\partial} + \bar{\partial}^\dagger$ ($\ker L = \ker L^2$ for any linear operator $L = L^*$). Being $(\bar{\partial} + \bar{\partial}^\dagger)(\omega)$ the sum of two pieces of different degree, both must vanish for $(\bar{\partial} + \bar{\partial}^\dagger)(\omega)$ to be zero: hence, ω is harmonic if and only if $\bar{\partial}\omega = \bar{\partial}^\dagger\omega = 0$.

Proposition 7.1 *For all k , there is an orthogonal decomposition*

$$\Omega^{0,k}(\mathbb{CP}_q^n) = \mathfrak{H}^{0,k}(\mathbb{CP}_q^n) \oplus \bar{\partial} \Omega^{0,k-1}(\mathbb{CP}_q^n) \oplus \bar{\partial}^\dagger \Omega^{0,k+1}(\mathbb{CP}_q^n) . \quad (24)$$

In particular, there is exactly one harmonic form for each cohomology class:

$$H_{\bar{\partial}}^k(\mathbb{CP}_q^n) \simeq \mathfrak{H}^{0,k}(\mathbb{CP}_q^n) .$$

Proof. With the inner product in (7), given two forms ω_1, ω_2 of degree $k-1$ and $k+1$ respectively, we have that

$$\langle \bar{\partial}\omega_1, \bar{\partial}^\dagger\omega_2 \rangle = \langle \bar{\partial}^2\omega_1, \omega_2 \rangle = 0 .$$

Thus, the spaces $\bar{\partial}\Omega^{0,k-1}(\mathbb{CP}_q^n)$ and $\bar{\partial}^\dagger\Omega^{0,k+1}(\mathbb{CP}_q^n)$ are orthogonal subspaces of $\Omega^{0,k}(\mathbb{CP}_q^n)$. It remains to show that a $(0,k)$ -form η is orthogonal to both $\bar{\partial}\Omega^{0,k-1}(\mathbb{CP}_q^n)$ and $\bar{\partial}^\dagger\Omega^{0,k+1}(\mathbb{CP}_q^n)$ if and only if it is harmonic. This follows from non-degeneracy of the inner product (i.e. from the faithfulness of the Haar state). We have:

$$\langle \eta, \bar{\partial}\omega_1 \rangle = \langle \bar{\partial}^\dagger\eta, \omega_1 \rangle = 0 , \quad \langle \eta, \bar{\partial}^\dagger\omega_2 \rangle = \langle \bar{\partial}\eta, \omega_2 \rangle = 0 ,$$

for all $\omega_1 \in \Omega^{0,k-1}(\mathbb{CP}_q^n)$ and $\omega_2 \in \Omega^{0,k+1}(\mathbb{CP}_q^n)$ if and only if $\bar{\partial}\eta = \bar{\partial}^\dagger\eta = 0$, that is if and only if η is harmonic. This establishes the decomposition in (24).

Forms in the subspace $\mathfrak{H}^{0,k}(\mathbb{CP}_q^n) \oplus \bar{\partial}\Omega^{0,k-1}(\mathbb{CP}_q^n)$ are $\bar{\partial}$ -closed by construction. On the other hand, a $\bar{\partial}$ -closed form $\omega \in \bar{\partial}^\dagger\Omega^{0,k+1}(\mathbb{CP}_q^n)$ must be harmonic (since $(\bar{\partial}^\dagger)^2 = 0$), and by orthogonality of the decomposition it must be zero. It follows that

$$H_{\bar{\partial}}^k(\mathbb{CP}_q^n) = \{ \mathfrak{H}^{0,k}(\mathbb{CP}_q^n) \oplus \bar{\partial}\Omega^{0,k-1}(\mathbb{CP}_q^n) \} / \bar{\partial}\Omega^{0,k-1}(\mathbb{CP}_q^n) = \mathfrak{H}^{0,k}(\mathbb{CP}_q^n) ,$$

and this concludes the proof. ■

We now compute $H_{\bar{\partial}}^k(\mathbb{CP}_q^n) \simeq \mathfrak{H}^{0,k}(\mathbb{CP}_q^n) = \ker \Delta_{\bar{\partial}}|_{\Omega^{0,k}(\mathbb{CP}_q^n)}$.

Proposition 7.2 *The Dolbeault cohomology groups of \mathbb{CP}_q^n are given by:*

$$H_{\bar{\partial}}^0(\mathbb{CP}_q^n) = \mathbb{C} , \quad H_{\bar{\partial}}^k(\mathbb{CP}_q^n) = 0 \quad \forall 1 \leq k \leq n .$$

Proof. Lemma 6.3 of [19], for $N = 0$ and with ℓ replaced by n , gives:

$$\Delta_{\bar{\partial}}\omega = \omega \triangleleft \left(\sum_{i=1}^n q^{-2i} X_i X_i^* + q^{-n-k} [k][n+1] \right) ,$$

for any $\omega \in \Omega^{0,k}(\mathbb{CP}_q^n)$, and with X_i the elements making up the operator $\bar{\partial}$ as in (22). Since the right hand side is a sum of two positive operators, $\Delta_{\bar{\partial}}\omega = 0$ if and only if one has both $\sum_{i=1}^n q^{-2i}\omega \triangleleft X_i X_i^* = 0$ and $[k][n+1] = 0$. The latter condition implies $H_{\bar{\partial}}^k(\mathbb{CP}_q^n) = \ker \Delta_{\bar{\partial}}|_{\Omega^{0,k}(\mathbb{CP}_q^n)} = 0$ for any $k \neq 0$.

For the remaining $k = 0$ case, Lemma 6.5 of [19], for $N = k = 0$, gives:

$$\triangleleft \left(\mathcal{C}_q - \sum_{i=1}^n q^{n+1-2i} X_i X_i^* \right) \Big|_{\Omega^{0,0}(\mathbb{CP}_q^n)} = \frac{q^{-n} + q[n] - [n+1]}{(q - q^{-1})^2} = 0 .$$

Therefore, for any $\omega \in \Omega^{0,0}(\mathbb{CP}_q^n)$ it holds that

$$q^{n+1} \Delta_{\bar{\partial}} \omega = \omega \triangleleft \mathcal{C}_q = \mathcal{C}_q \triangleright \omega ,$$

the second equality following from the fact that for central elements the left and right canonical actions coincide (cf. the proof of Lemma 3.1 of [18]).

Now, as left $U_q(\mathfrak{su}(n+1))$ -modules, [19, Prop. 5.5] yields the equivalence:

$$\Omega^{0,0}(\mathbb{CP}_q^n) \simeq \bigoplus_{m \in \mathbb{N}} V_{(m,0,\dots,0,m)} ,$$

where $V_{(m,0,\dots,0,m)}$ is the vector space carrying the irreducible representation of highest weight $(m, 0, \dots, 0, m)$. Finally, from [19, Prop. 3.3] the restriction of \mathcal{C}_q to this representation is $[m][m+n]$ times the identity operator, and vanishes if and only if $m = 0$. Thus, $H_{\bar{\partial}}^0(\mathbb{CP}_q^n) = V_{(0,0,\dots,0)} \simeq \mathbb{C}$ coincides with the trivial representation. This concludes the proof. \blacksquare

7.3 Holomorphic modules

As in §6.4, a holomorphic connection on a $\mathcal{A}(\mathbb{CP}_q^n)$ -module \mathcal{E} is a connection associated to the calculus $(\Omega^{0,\bullet}(\mathbb{CP}_q^n), \bar{\partial})$. For the modules $\mathcal{E} = \Gamma_N$ such a connection, that here we denote by $\nabla_N^{\bar{\partial}}$, was given in [19, Sect. 6]. Indeed, as discussed in [39, Sect. 5] these are bimodule connections, and, using their isomorphism $\lambda_N : \Gamma_N \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \Omega^{0,1} \rightarrow \Omega^{0,1} \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \Gamma_N$, one passes from the left to the right version. We need a preliminary lemma.

Lemma 7.3 *For any $\eta \in \Gamma_N$, $\nabla_N^{\bar{\partial}} \eta$ is the vector with components*

$$(\nabla_N^{\bar{\partial}} \eta)_i = q^{\frac{N}{2}-1} \eta \triangleleft F_n F_{n-1} \dots F_i , \quad i = 1, \dots, n . \quad (25)$$

Proof. By [19, Lemma 6.1], the connection $\nabla_N^{\bar{\partial}}$ coincides with the operator $\bar{\partial}$ on Γ_N . In turn, since $\eta \in \Gamma_N$ is by definition in the kernel of the right action of $U_q(\mathfrak{su}(n))$ while $\eta \triangleleft K_n^{-1} = q^{N/2} \eta$, the vector $\bar{\partial} \eta$ has components

$$\begin{aligned} (\bar{\partial} \eta)_i &= \eta \triangleleft S^{-1}(\hat{K} X_i) = \eta \triangleleft S^{-1}(\hat{K} N_{i,n} M_{i,n}^*) \\ &= \eta \triangleleft S^{-1}(K_i K_{i+1} \dots K_n M_{i,n}^*) = q^{-1} \eta \triangleleft S^{-1}(M_{i,n}^* K_i K_{i+1} \dots K_n) \\ &= q^{-1} \eta \triangleleft K_i^{-1} K_{i+1}^{-1} \dots K_n^{-1} S^{-1}(M_{i,n}^*) = q^{\frac{N}{2}-1} \eta \triangleleft S^{-1}(M_{i,n}^*) , \end{aligned}$$

with $X_i = N_{i,n} M_{i,n}^*$ as in (23). Being $x \mapsto S^{-1}(x^*)$ an algebra morphism,

$$S^{-1}(M_{i,n}^*) = [S^{-1}(E_i^*), S^{-1}(M_{i+1,n}^*)]_q = -q[F_i, S^{-1}(M_{i+1,n}^*)]_q$$

$$= -qF_i S^{-1}(M_{i+1,n}^*) + S^{-1}(M_{i+1,n}^*)F_i .$$

Since $\eta \triangleleft F_i = 0$ for all $i \neq n$, we get:

$$\begin{aligned} (\bar{\partial}\eta)_i &= q^{\frac{N}{2}-1} \eta \triangleleft S^{-1}(M_{i,n}^*) = q^{\frac{N}{2}-1} \eta \triangleleft S^{-1}(M_{i+1,n}^*)F_i \\ &= q^{\frac{N}{2}-1} \eta \triangleleft S^{-1}(M_{i+2,n}^*)F_{i+1}F_i = \dots = q^{\frac{N}{2}-1} \eta \triangleleft F_n F_{n-1} \dots F_i . \end{aligned}$$

This concludes the proof. ■

(For $N = 0$, eq. (13) of [39] has an extra minus sign and misses a factor q^{-1} .)

It follows from Lemma 6.1 and Prop. 5.6 of [19] that the connection is flat, $(\nabla_N^{\bar{\partial}})^2 = 0$. For the corresponding space of holomorphic sections $H^0(\Gamma_N, \nabla_N^{\bar{\partial}})$ we have next proposition. This is essentially Cor. 4.2 of [39], of which we give an easier proof (also filling few gaps). With notations $k_{N,n} = \binom{|N|+n}{n}$;

Proposition 7.4 *The cohomology groups are: $H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = 0$ if $N > 0$, and $H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) \simeq \mathbb{C}^{k_{N,n}}$ if $N \leq 0$. Explicitly, for any $N \leq 0$, $H^0(\Gamma_N, \nabla_N^{\bar{\partial}})$ is the \mathbb{C} -space of degree $|N|$ polynomials in the z_i 's.*

Proof. By definition, and using (25):

$$H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = \{\eta \in \Gamma_N : \nabla_N^{\bar{\partial}} \eta = 0\} = \{\eta \in \Gamma_N : \eta \triangleleft F_n = 0\} .$$

Recall that the relations between ‘coordinates’ on S_q^{2n+1} and generators of $\mathcal{A}(SU_q(n+1))$ is $z_i = u_{n+1-i}^{n+1}$, while from [19, eq. (4.1)], $\pi_i^{n+1}(F_j) = \pi_{n+1}^i(E_j)^* = 0$ for all $j = 1, \dots, n$. Hence $z_i \triangleleft F_j = \sum_k \pi_i^{n+1}(F_j) u_i^k = 0$ for all indices i, j . Recall also that the elements ψ_{j_0, \dots, j_n}^N in (8b) are a generating family of Γ_N . For $N \leq 0$ they are degree $|N|$ monomials in the z_i 's, hence they are in the kernel of F_n . Thus,

$$\psi_{j_0, \dots, j_n}^N \in H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) \quad \forall N \leq 0 .$$

Since for a fixed N the elements ψ_{j_0, \dots, j_n}^N are independent over \mathbb{C} , and their number is $k_{N,n} = \binom{|N|+n}{n}$, we also have $\dim H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) \geq k_{N,n}$ for all $N \leq 0$. We show next that $\dim H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = k_{N,n}$ if $N \leq 0$ and that $\dim H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = 0$ if $N > 0$, thus concluding the proof.

From [19, Prop. 5.5] we read the decomposition of Γ_N ,

$$\Gamma_N = \Omega_N^0 \simeq \begin{cases} \bigoplus_{m \geq 0} V_{(m+N, 0, \dots, 0, m)} , & \text{if } N > 0 , \\ \bigoplus_{m \geq -N} V_{(m+N, 0, \dots, 0, m)} , & \text{if } N \leq 0 \end{cases}$$

into irreducible representations, with the highest weight vector $v_{m,N}$ of the representation $V_{(m+N,0,\dots,0,m)}$ explicitly given by

$$v_{m,N} := z_0^m (z_n^*)^{m+N} = \begin{cases} (p_{0n})^m \psi_{0,\dots,0,N}^N & \text{if } N > 0, \\ \psi_{-N,0,\dots,0}^N (p_{0n})^{m+N} & \text{if } N \leq 0. \end{cases}$$

Indeed $v_{m,N} \in \Gamma_N$, and using the formulæ for the left action in (11) (remembering that $z_i = z'_{n+1-i}$), one checks that $E_i \triangleright v_{m,N} = 0$, for all $i = 1, \dots, n$, i.e. $v_{m,N}$ is a highest weight vector, and $K_i \triangleright v_{m,N} = q^{\frac{1}{2}(m+N)\delta_{i,1} + \frac{1}{2}m\delta_{i,n}} v_{m,N}$, i.e. its weight is $(m+N, 0, \dots, 0, m)$ as claimed.

Let $T_{m,N}$ be the restriction of $\nabla_N^{\bar{\partial}}$ to the subspace $V_{(m+N,0,\dots,0,m)}$ of Γ_N . Since left and right canonical actions commute, the image of $T_{m,N}$ is a copy of the same representation $V_{(m+N,0,\dots,0,m)}$ inside $\Omega^{0,1}(\mathbb{CP}_q^n) \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \Gamma_N$. For the same reason, $\ker T_{m,N}$ carries a representation of $U_q(\mathfrak{su}(n+1))$. For fixed N , each $T_{m,N}$ has distinct domain and image, hence $\nabla_N^{\bar{\partial}} = \bigoplus_m T_{m,N}$ and

$$H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = \bigoplus_m \ker T_{m,N}.$$

Being $\ker T_{m,N}$ a representation of $U_q(\mathfrak{su}(n+1))$, it is either the whole $V_{(m+N,0,\dots,0,m)}$ or it is $\{0\}$, since the representation $V_{(m+N,0,\dots,0,m)}$ is irreducible. To discern among the two possibilities, it is enough to check whether or not $v_{m,N}$ is in the kernel of $T_{m,N}$. Using $z_i \triangleleft F_n = 0$ and $z_i \triangleleft K_n = q^{\frac{1}{2}z_i}$ one finds

$$v_{m,N} \triangleleft F_n = q^{-\frac{m}{2}} z_0^m \{ (z_n^*)^{m+N} \triangleleft F_n \}.$$

Using $z_i \triangleleft E_n = u_{n+1-i}^n$, one finds

$$\begin{aligned} (z_n)^{m+N} \triangleleft E_n &= \sum_{k=0}^{m+N-1} q^{\frac{1}{2}(m+N-2k-1)} (z_n)^k u_1^n (z_n)^{m+N-k-1} \\ &= u_1^n (z_n)^{m+N-1} q^{\frac{1}{2}(m+N-1)} \sum_{k=0}^{m+N-1} q^{-2k} \\ &= u_1^n (z_n)^{m+N-1} q^{-\frac{1}{2}(m+N-1)} [m+N]. \end{aligned}$$

Since $a^* \triangleleft F_n = -q^{-1}(a \triangleleft E_n)^*$, this finally results into

$$v_{m,N} \triangleleft F_n = -q^{-m-\frac{1}{2}(N+1)} [m+N] z_0^m (z_n^*)^{m+N-1} (u_1^n)^*,$$

and this is zero if and only if $m+N=0$.

Thus $\ker T_{m,N} \neq \{0\}$ if and only if $m = -N$, admissible only if $N \leq 0$. If $N \leq 0$, then $H^0(\Gamma_N, \nabla_N^{\bar{\partial}}) = \ker T_{-N,N} = V_{(0,\dots,0,-N)}$. By (3.15) of [19] its dimension is given by

$$\frac{\prod_{1 \leq r \leq s \leq n} (s - r + 1)}{\prod_{r=1}^n r!} \prod_{1 \leq r \leq s=n} (s - r + 1 - N) = \frac{\prod_{s=1}^{n-1} s! (-N + n)!}{\prod_{r=1}^n r! (-N)!} = \binom{-N + n}{n},$$

and this concludes the proof. \blacksquare

From last proposition there is a vector space isomorphism

$$\bigoplus_{N \leq 0} H^0(\Gamma_N, \nabla_N^{\bar{\theta}}) \simeq \frac{\mathbb{C} \langle z_0, \dots, z_n \rangle}{\langle z_i z_j - q^{-1} z_j z_i, 0 \leq i < j \leq n \rangle}. \quad (26)$$

The right hand side is also a ring (in fact, it is a complex unital algebra, although not a $*$ -algebra), called the “quantum homogeneous coordinate ring” starting with the paper [37] for $n = 1$. The isomorphism (26) becomes an isomorphism of graded unital algebras if we endow the left hand side with the product induced by tensor product of bimodules. From [39, Prop. 5.2] the product of holomorphic sections is a holomorphic section, a fact also inferable from the explicit expression of the isomorphism $\Gamma_N \otimes_{\Gamma_0} \Gamma_M \rightarrow \Gamma_{N+M}$. This we show now, for the sake of completeness. Recall that $\Gamma_0 = \mathcal{A}(\mathbb{CP}_q^n)$.

Lemma 7.5 *For any $N, M \in \mathbb{Z}$, it holds that $\Gamma_N \otimes_{\Gamma_0} \Gamma_M \simeq \Gamma_{M+N}$.*

Proof. It is enough to prove that a) $\Gamma_N \otimes_{\Gamma_0} \Gamma_1 \simeq \Gamma_{N+1}$ for all $N \in \mathbb{Z}$, b) $\Gamma_1 \otimes_{\Gamma_0} \Gamma_{-1} \simeq \Gamma_0$. Indeed, from a) and b) it follows that $\Gamma_{N+1} \otimes_{\Gamma_0} \Gamma_{-1} \simeq \Gamma_N$, and with this one proves that $\Gamma_N \otimes_{\Gamma_0} \Gamma_M \simeq \Gamma_{M+N}$ by induction on M . A bimodule map $m : \Gamma_N \otimes_{\Gamma_0} \Gamma_M \rightarrow \Gamma_{M+N}$ is given by the multiplication. We now define bimodule maps $\Gamma_{M+N} \rightarrow \Gamma_N \otimes_{\Gamma_0} \Gamma_M$ in the cases a) and b), and prove they are inverse maps to m . Define:

$$\begin{aligned} \phi : \Gamma_{N+1} &\rightarrow \Gamma_N \otimes_{\Gamma_0} \Gamma_1, & \phi(\eta) &= \sum_{k=0}^n \eta z_k \otimes z_k^*, \\ \chi : \Gamma_0 &\rightarrow \Gamma_1 \otimes_{\Gamma_0} \Gamma_{-1}, & \chi(a) &= \sum_{k=0}^n q^{2k} a z_k^* \otimes z_k. \end{aligned}$$

A straightforward computation shows that $\sum_k z_k \otimes z_k^*$ and $\sum_k q^{2k} z_k^* \otimes z_k$ commute with the generators of Γ_0 , so that ϕ and χ are bimodule maps. Moreover:

$$\begin{aligned} (m \circ \phi)(\eta) &= m \circ \left(\sum_k \eta z_k \otimes z_k^* \right) = \eta \sum_k z_k z_k^* = \eta, \\ (m \circ \chi)(a) &= m \circ \left(\sum_k q^{2k} a z_k^* \otimes z_k \right) = a \sum_k q^{2k} z_k^* z_k = a, \end{aligned}$$

so m is a left inverse of both ϕ and χ (remember that $\sum_k q^{2k} z_k^* z_k = 1$).

Being the elements z_i^* a generating family of Γ_1 , and the elements z_i a generating family of Γ_{-1} , any $\eta \in \Gamma_N \otimes_{\Gamma_0} \Gamma_1$ can be written as $\eta = \sum_i \eta_i \otimes z_i^*$ with $\eta_i \in \Gamma_N$, and any $\xi \in \Gamma_1 \otimes_{\Gamma_0} \Gamma_{-1}$ can be written as $\xi = \sum_i \xi_i \otimes z_i$ with $\xi_i \in \Gamma_1$. A simple computation yields

$$\begin{aligned} (\phi \circ m)(\eta) &= \phi \left(\sum_i \eta_i z_i^* \right) = \sum_{i,k} \eta_i z_i^* z_k \otimes z_k^* = \sum_{i,k} \eta_i \otimes z_i^* z_k z_k^* \\ &= \sum_i \eta_i \otimes z_i^* = \eta, \\ (\chi \circ m)(\xi) &= \chi \left(\sum_i \xi_i z_i \right) = \sum_{i,k} q^{2k} \xi_i z_i z_k^* \otimes z_k = \sum_{i,k} \xi_i \otimes q^{2k} z_i z_k^* z_k \\ &= \sum_i \xi_i \otimes z_i = \xi, \end{aligned}$$

having used the fact that $z_i^* z_k$ and $z_i z_k^*$ belong to Γ_0 to move them to the right hand side of the tensor product. Thus m is also a right inverse of both ϕ and χ . It is then an isomorphism of bimodules. ■

7.4 Existence of a twisted positive Hochschild cocycle

For a closed oriented Riemannian manifold M of real dimension $2n$, one defines a Hochschild $2n$ -cocycle τ , the “fundamental class” of M [7, Sect. VI.2], as

$$\tau(a_0, \dots, a_{2n}) := \int_M a_0 da_1 \wedge da_2 \wedge \dots \wedge da_{2n}. \quad (27)$$

If M is a complex manifold, a representative of the class $[\tau]$ is:

$$\int_M a_0 \partial a_1 \wedge \dots \wedge \partial a_n \wedge \bar{\partial} a_{n+1} \wedge \dots \wedge \bar{\partial} a_{2n}, \quad (28)$$

modulo a proportionality constant that here we neglect. The latter is a *positive* cocycle in the sense of [10]. It is worth stressing that (27) is also cyclic while (28) is not. Positive representatives of $[\tau]$ form a convex space and, for $n = 1$, there is a bijection between its extreme points and complex structures on M (cf. §VI.2 of [7]). Another way to construct positive representatives of $[\tau]$ uses the Clifford representation of differential forms, cf. [10, Sect. IV, Example 3], leading to a cocycle that depends only on the conformal class of the Riemannian metric (in complex dimension 1, conformal and complex structures are equivalent).

For the case of \mathbb{CP}_q^n , having the Dolbeault complex, the next step would be to construct a full differential $*$ -calculus, reducing to the de Rham complex

for $q = 1$. For $n = 2$ this was done explicitly in [22]. In [39, Sect. 6], a positive representative of their fundamental twisted Hochschild cocycle was given under the (hidden) assumption that there is a product of forms with the property that $\Omega^{n,n}(\mathbb{CP}_q^n) \simeq \mathcal{A}(\mathbb{CP}_q^n)$ is a free bimodule of rank 1, i.e. there exists a basis element of $\Omega^{n,n}(\mathbb{CP}_q^n)$ that one would take as a “volume form”. At an algebraic level (i.e. without using operators on Hilbert spaces) such a calculus was given in [35], where the existence of a volume form was also established. One should stress that it is not clear whether or not the $*$ -calculus in [35] is related to the Dolbeault complex discussed here, although it is reasonable to guess that, modulo isomorphisms, the former is an extension of the latter.

We now show that in fact the Dolbeault complex is enough to define a positive twisted Hochschild $2n$ -cocycle (although one would need a full $*$ -calculus in order to have an analogue of (27)). We will give a positive $2k$ -cocycle for any $0 \leq k \leq n$. Let us recall some basic facts and definitions.

Let \mathcal{A} be a $*$ -algebra and η an automorphism of \mathcal{A} (not a $*$ -automorphism: we do not assume $\eta(a^*) = \eta(a)^*$). We denote by ${}_{\eta}\mathcal{A}$ the \mathcal{A} -bimodule that is \mathcal{A} itself as a vector space and as a right module, but has a left module structure ‘twisted’ with η : $(a, b) \mapsto \eta(a)b$ for $a, b \in \mathcal{A}$. The Hochschild cohomology $HH_{\eta}^{\bullet}(\mathcal{A}) = H^{\bullet}(\mathcal{A}, {}_{\eta}\mathcal{A})$ is the cohomology of the complex $(\text{Hom}_{\mathbb{C}}(\mathcal{A}^{\bullet}, \mathbb{C}), b_{\bullet})$, where the coboundary operators b is [45]:

$$\begin{aligned} b\varphi(a_0, a_1, \dots, a_k) &= \sum_{i=0}^{k-1} (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_k) \\ &\quad + (-1)^k \varphi(\eta(a_k) a_0, a_1, \dots, a_{k-1}). \end{aligned}$$

The cocycle φ is called *positive* if the sesquilinear form on \mathcal{A}^{n+1} given by

$$\langle a_0 \otimes a_1 \otimes \dots \otimes a_n, b_0 \otimes b_1 \otimes \dots \otimes b_n \rangle_{\varphi} := \varphi(\eta(b_0^*) a_0, a_1, \dots, a_n, b_n^*, \dots, b_1^*)$$

is positive semidefinite.

Remark 7.6 In the original definition one assumes that $\langle \cdot, \cdot \rangle_{\varphi}$ is positive definite. But already for (28), for $n \geq 2$, this is not true since for example elements $a \otimes a \otimes \dots \otimes a$ are not zero in \mathcal{A}^{n+1} , but $\partial a \wedge \partial a = 0$. Even for $n = 1$, $\langle \cdot, \cdot \rangle_{\varphi}$ is only positive definite on $\mathcal{A} \otimes (\mathcal{A}/\mathbb{C})$. Similarly, looking at the proof of [39, Thm. 6.1], it is clear that the cocycle φ there is only positive semidefinite, since the Haar state is faithful, but the map $\mathcal{A}^{n+1} \rightarrow \Omega^{0,n}$, $a_0 \otimes a_1 \otimes \dots \otimes a_n \mapsto a_0 \partial a_1 \dots \partial a_n$, is not injective.

For \mathbb{CP}_q^n , one defines non-trivial positive twisted Hochschild cocycles with η the inverse of the modular automorphism, that is:

$$\eta(a) = K_{2\rho}^{-1} \triangleright a, \quad \forall a \in \mathcal{A}(\mathbb{CP}_q^n),$$

with $K_{2\rho}$ the element, implementing the square of the antipode, given in (13). From [19, eq. (3.4)] and right invariance of elements of $\mathcal{A}(\mathbb{CP}_q^n)$, it follows that the Haar state h is the representative of an element in $HH_\eta^0(\mathcal{A}(\mathbb{CP}_q^n))$, that is

$$h(ab) = h(\eta(b)a), \quad \forall a, b \in \mathcal{A}(\mathbb{CP}_q^n). \quad (29)$$

Define $\tau_0 := h$ and, for any $1 \leq k \leq n$, define a $2k$ -cochain τ_k on \mathbb{CP}_q^n by

$$\begin{aligned} \tau_k(a_0, a_1, \dots, a_{2k}) &= \langle \bar{\partial}a_k^* \wedge_q \dots \wedge_q \bar{\partial}a_1^* a_0^*, \bar{\partial}a_{k+1} \wedge_q \dots \wedge_q \bar{\partial}a_{2k} \rangle_{\Omega^{0,k}(\mathbb{CP}_q^n)} \\ &= h\left(a_0(\bar{\partial}a_k^* \wedge_q \dots \wedge_q \bar{\partial}a_1^*)^* \cdot (\bar{\partial}a_{k+1} \wedge_q \dots \wedge_q \bar{\partial}a_{2k})\right). \end{aligned} \quad (30)$$

In the first line we have the canonical inner product of $\Omega^{0,k}(\mathbb{CP}_q^n)$, as given in §3.1. Elements $\omega_1, \omega_2 \in \Omega^{0,k}(\mathbb{CP}_q^n)$ are column vectors with $\binom{n}{k}$ components and with entries in $\mathcal{A}(SU_q(n+1))$; ω_1^* is the transposed conjugated row vector, and the product $\omega_1^* \cdot \omega_2$ in (30) is the row-by-column product composed with the multiplication in $\mathcal{A}(SU_q(n+1))$.

Remark 7.7 For $q = 1$, τ_n in (30) coincides with (28) modulo a sign: the property $(\bar{\partial}a^*)^* = \partial a$ yields

$$\tau_n(a_0, \dots, a_{2n}) = (-1)^{\frac{n(n-1)}{2}} \int_{\mathbb{CP}^n} a_0 \eta_1 \eta_2,$$

where $\eta_1 = \partial a_1 \wedge \dots \wedge \partial a_n \in \Omega^{n,0}(\mathbb{CP}^n) \simeq \Gamma_{n+1}$ and $\eta_2 = \bar{\partial}a_{n+1} \wedge \dots \wedge \bar{\partial}a_{2n} \in \Omega^{0,n}(\mathbb{CP}^n) \simeq \Gamma_{-n-1}$ are scalar function on $SU(n+1)$, and the product of $(n, 0)$ forms with $(0, n)$ forms is simply the product of the corresponding functions. The integral is normalized so that $\int_{\mathbb{CP}^n} 1 = 1$.

Proposition 7.8 *The map τ_k in (30) is a positive representative of an element $[\tau_k] \in HH_\eta^{2k}(\mathcal{A}(\mathbb{CP}_q^n))$.*

Proof. From (29) it follows that

$$\begin{aligned} &\langle a_0 \otimes a_1 \otimes \dots \otimes a_k, b_0 \otimes b_1 \otimes \dots \otimes b_k \rangle_{\tau_k} \\ &= \langle \bar{\partial}a_k^* \wedge_q \dots \wedge_q \bar{\partial}a_1^* a_0^*, (\bar{\partial}b_k^* \wedge_q \dots \wedge_q \bar{\partial}b_1^*) b_0^* \rangle_{\Omega^{0,k}(\mathbb{CP}_q^n)}. \end{aligned}$$

This is positive semidefinite, since $\langle \omega, \omega \rangle_{\Omega^{0,k}(\mathbb{CP}_q^n)} \geq 0$ for all $\omega \in \Omega^{0,k}(\mathbb{CP}_q^n)$. Let us write

$$\tau_k(a_0, a_1, \dots, a_{2k}) = h\left(a_0 \phi(a_1, \dots, a_k) \cdot \phi'(a_{k+1}, \dots, a_{2k})\right),$$

with

$$\phi(a_1, \dots, a_k) := (\bar{\partial} a_k^* \wedge_q \dots \wedge_q \bar{\partial} a_1^*)^* , \quad \phi'(b_1, \dots, b_k) := \bar{\partial} b_1 \wedge_q \dots \wedge_q \bar{\partial} b_k ,$$

and recall that the product between ϕ and ϕ' is the row-by-column product between two vectors with entries in $\mathcal{A}(SU_q(n+1))$. Using the Leibniz rule and the rule for the involution, that is $\bar{\partial}(a_i a_{i+1})^* = a_{i+1}^* (\bar{\partial} a_i^*) + (\bar{\partial} a_{i+1}^*) a_i^*$, we compute

$$\begin{aligned} \sum_{i=1}^k (-1)^i \phi(a_1, \dots, a_i a_{i+1}, \dots, a_{k+1}) \\ = (-1)^1 a_1 \phi(a_2, \dots, a_{k+1}) + (-1)^k \phi(a_1, \dots, a_k) a_{k+1} , \end{aligned} \quad (31)$$

and

$$\begin{aligned} \sum_{i=k+1}^{2k} (-1)^i \phi'(a_{k+1}, \dots, a_i a_{i+1}, \dots, a_{2k+1}) \\ = (-1)^{k+1} a_{k+1} \phi'(a_{k+2}, \dots, a_{2k+1}) + (-1)^{2k} \phi'(a_{k+1}, \dots, a_{2k}) a_{2k+1} . \end{aligned} \quad (32)$$

Therefore,

$$\begin{aligned} b\tau_k(a_0, \dots, a_{2k+1}) &= h \left(a_0 a_1 \phi(\dots) \phi'(\dots) \right. \\ &\quad - a_0 a_1 \phi(\dots) \phi'(\dots) + (-1)^k a_0 \phi(\dots) a_{k+1} \phi'(\dots) \\ &\quad + (-1)^{k+1} a_0 \phi(\dots) a_{k+1} \phi'(\dots) + a_0 \phi(\dots) \phi'(\dots) a_{2k+1} \\ &\quad \left. - \eta(a_{2k+1}) a_0 \phi(\dots) \phi'(\dots) \right) . \end{aligned}$$

having used (31) for the second line and (32) for the third line. We can simplify the first four terms and get:

$$b\tau_k(a_0, \dots, a_{2k+1}) = h \left(a_0 \phi(\dots) \phi'(\dots) a_{2k+1} - \eta(a_{2k+1}) a_0 \phi(\dots) \phi'(\dots) \right) ,$$

that is zero by (29). ■

7.5 Quantum characteristic classes

A natural map from equivariant K-theory to equivariant cyclic homology is given in [49] (among others), and adapted to the present situation in [22]. As explained in Sect. 7.1 of the latter, equivariant cyclic homology is paired with twisted Hochschild homology, inducing a pairing,

$$HH_{\eta}^{\bullet}(\mathcal{A}(\mathbb{CP}_q^n)) \times K_0^{U_q(\mathfrak{su}(n+1))}(\mathcal{A}(\mathbb{CP}_q^n)) \rightarrow \mathbb{C} ,$$

of which here we just give the formula. For a representation $\sigma : U_q(\mathfrak{su}(n+1)) \rightarrow M_k(\mathbb{C})$, an idempotent $p \in M_k(\mathcal{A}(\mathbb{CP}_q^n))$ satisfying (10), and a twisted cycle $\tau \in HH_{\eta}^m(\mathcal{A}(\mathbb{CP}_q^n))$, one has,

$$\langle [\tau], [(p, \sigma)] \rangle = \tau \left(\text{Tr}_{\mathbb{C}^k} \left(\overbrace{p \dot{\otimes} p \dot{\otimes} \dots \dot{\otimes} p}^{m+1 \text{ times}} \sigma(K_{2\rho}^{-1})^t \right) \right) ,$$

with $\dot{\otimes}$ composition of tensor product over \mathbb{C} with matrix multiplication.

Proposition 7.9 *For any $N \in \mathbb{Z}$ and any $0 \leq k \leq n$, with τ_k the cocycle in (30) and (P'_N, σ^N) the element in §3.2 one has:*

$$\langle [\tau_k], [(P'_N, \sigma^N)] \rangle = \begin{cases} 1 & \text{if } k = 0 , \\ q^{-n-3}[n][N] & \text{if } k = 1 , \\ 0 & \text{if } k \geq 2 . \end{cases}$$

Proof. Recall that $P'_N = R_N P_N R_N^{-1}$, with $R_N = \sigma^N(K_{2\rho})^t$ and $P_N = \Psi_N \Psi_N^{\dagger}$ the projections in §3.2. Since R_N is a constant matrix and the trace is cyclic:

$$\langle [\tau_k], [(P'_N, \sigma^N)] \rangle = \tau_k \left(\text{Tr} \left(P_N \dot{\otimes} P_N \dot{\otimes} \dots \dot{\otimes} P_N \sigma(K_{2\rho}^{-1})^t \right) \right) ,$$

so that there is no difference in using P_N or P'_N . Since $P_N^* = P_N$:

$$\begin{aligned} & \langle [\tau_k], [(P'_N, \sigma^N)] \rangle \\ &= h \left(\text{Tr} \left(P_N (\bar{\partial} P_N \dot{\wedge}_q \dots \dot{\wedge}_q \bar{\partial} P_N)^* (\bar{\partial} P_N \dot{\wedge}_q \dots \dot{\wedge}_q \bar{\partial} P_N) \sigma(K_{2\rho}^{-1})^t \right) \right) , \end{aligned}$$

with $\dot{\wedge}_q$ the composition of $\dot{\wedge}_q$ with matrix multiplication. Using $P_N = \Psi_N \Psi_N^{\dagger}$, of [19, eq. (3.4)], cyclicity of the trace, $\Psi_N \triangleleft K_{2\rho} = \Psi_N \triangleleft K_n^{2n} = q^{-nN} \Psi_N$ and $K_{2\rho} \triangleright \Psi_N = \sigma(K_{2\rho})^t \Psi_N$ — cf. (12) — we get

$$\begin{aligned} & \langle [\tau_k], [(P'_N, \sigma^N)] \rangle \\ &= q^{-nN} h \left(\Psi_N^{\dagger} (\bar{\partial} P_N \dot{\wedge}_q \dots \dot{\wedge}_q \bar{\partial} P_N)^* (\bar{\partial} P_N \dot{\wedge}_q \dots \dot{\wedge}_q \bar{\partial} P_N) \Psi_N \right) . \end{aligned}$$

From the Leibniz rule and $\Psi_N^{\dagger} = \Psi_N^{\dagger} P_N$ it follows $(\bar{\partial} P_N) P_N = (1 - P_N) \bar{\partial} P_N$. From this and $\Psi_N = P_N \Psi_N$ it follows

$$(\bar{\partial} P_N \dot{\wedge}_q \bar{\partial} P_N) \Psi_N = P_N (\bar{\partial} P_N \dot{\wedge}_q \bar{\partial} P_N) \Psi_N = \Psi_N (\nabla_N^{\bar{\partial}})^2 = 0 ,$$

since $\nabla_N^{\bar{\partial}}$ is flat. Thus, if $k \geq 2$ the pairing $\langle [\tau_k], [(P'_N, \sigma^N)] \rangle$ is zero.

If $k = 0$, $\langle [\tau_0], [(P'_N, \sigma^N)] \rangle = h(\Psi_N^\dagger \Psi_N) = h(1) = 1$.

The remaining case is $k = 1$. Now,

$$\langle [\tau_1], [(P'_N, \sigma^N)] \rangle = q^{-N} \sum_{i=1}^n h(\Psi_N^\dagger (\bar{\partial} P_N)_i^* (\bar{\partial} P_N)_i \Psi_N) ,$$

where $\bar{\partial} P_N$ is a matrix with entries $(0, 1)$ -forms, i.e. vectors with n components labelled by $i = 1, \dots, n$. From (25) (recall that $\bar{\partial} = \nabla_0^{\bar{\partial}}$) we get:

$$(\bar{\partial} P_N)_i = q^{-1} P_N \triangleleft F_n F_{n-1} \dots F_i .$$

The case $N < 0$ being similar, let us take $N \geq 0$. Then $\Psi_n^\dagger \triangleleft F_n = 0$, since $z_i \triangleleft F_n = 0$, and $\eta \triangleleft F_i = \eta \triangleleft E_i = 0$ for $\eta \in \Gamma_N$ and $i = 1, \dots, n-1$. Therefore

$$\begin{aligned} \bar{\partial}(\Psi_N \Psi_N^\dagger) &= q^{-1} (\Psi_N \triangleleft F_n F_{n-1} \dots F_i) (\Psi_N^\dagger \triangleleft K_n K_{n-1} \dots K_i) \\ &= q^{\frac{N}{2}-1} (\Psi_N \triangleleft F_n F_{n-1} \dots F_i) \Psi_N^\dagger . \end{aligned}$$

In turn, all of this yields:

$$\begin{aligned} \langle [\tau_1], [(P'_N, \sigma^N)] \rangle &= q^{-2} \sum_{i=1}^n h((\Psi_N \triangleleft F_n F_{n-1} \dots F_i)^* (\Psi_N \triangleleft F_n F_{n-1} \dots F_i)) \\ &= q^{-2} \sum_{i=1}^n q^{-2(n-i+1)} h((\mathcal{L}_{F_i \dots F_n} \Psi_N)^* (\mathcal{L}_{F_i \dots F_n} \Psi_N)) . \end{aligned}$$

From unitarity of the \mathcal{L} action:

$$\begin{aligned} \langle [\tau_1], [(P'_N, \sigma^N)] \rangle &= q^{-2} \sum_{i=1}^n q^{-2(n-i+1)} h(\Psi_N^\dagger (\mathcal{L}_{F_i \dots F_n}^* \mathcal{L}_{F_i \dots F_n} \Psi_N)) \\ &= q^{-2} \sum_{i=1}^n q^{-2(n-i+1)} h(\Psi_N^\dagger (\mathcal{L}_{E_n \dots E_i F_i \dots F_n} \Psi_N)) \\ &= q^{-2} \sum_{i=1}^n q^{-2(n-i+1)} h(\Psi_N^\dagger (\Psi_N \triangleleft F_n \dots F_i E_i \dots E_n)) . \end{aligned}$$

Since $\Psi_N \triangleleft E_i = 0$ for $i = 1, \dots, n$ and $[E_i, F_j] = 0$ when $i \neq j$, we have:

$$\begin{aligned} \Psi_N \triangleleft F_n \dots F_i E_i \dots E_n &= \Psi_N \triangleleft F_n \dots F_{i+1} [F_i, E_i] E_{i+1} \dots E_n \\ &= \Psi_N \triangleleft F_n \dots F_{i+1} \frac{K_i^{-2} - K_i^2}{q - q^{-1}} E_{i+1} \dots E_n \\ &= \Psi_N \triangleleft \frac{q K_i^{-2} - q^{-1} K_i^2}{q - q^{-1}} F_n \dots F_{i+1} E_{i+1} \dots E_n , \end{aligned}$$

having used in the last equality $K_i F_{i+1} K_i^{-1} = q^{\frac{1}{2}} F_{i+1}$ and $K_i F_j K_i^{-1} = F_j$ for $j > i + 1$ (the defining relations of $U_q(\mathfrak{su}(n+1))$). From $\Psi_N \triangleleft K_i = \Psi_N$, if $i < n$ we get:

$$\Psi_N \triangleleft F_n \dots F_i E_i \dots E_n = \Psi_N \triangleleft F_n \dots F_{i+1} E_{i+1} \dots E_n ,$$

and by induction on i :

$$\begin{aligned} \Psi_N \triangleleft F_n \dots F_i E_i \dots E_n &= \Psi_N \triangleleft F_n E_n = \Psi_N \triangleleft [F_n, E_n] \\ &= \Psi_N \triangleleft \frac{K_n^{-2} - K_n^2}{q - q^{-1}} = [N] \Psi_N . \end{aligned}$$

Therefore,

$$\begin{aligned} \langle [\tau_1], [(P'_N, \sigma^N)] \rangle &= q^{-2} [N] \sum_{i=1}^n q^{-2(n-i+1)} h(\Psi_N^\dagger \Psi_N) \\ &= q^{-2} [N] \sum_{i=1}^n q^{-2(n-i+1)} = q^{-n-3} [n] [N] . \end{aligned}$$

This concludes the proof. ■

As a consequence of previous proposition, the idempotents P'_N represent distinct elements in equivariant K-theory, since $\langle [\tau_1], [(P'_N, \sigma^N)] \rangle = \langle [\tau_1], [(P'_M, \sigma^M)] \rangle$ if and only if $N = M$. This is consistent with [55, Prop. 3.8], where it is shown that these idempotents generates $K_0^{U_q(\mathfrak{su}(2))}(\mathcal{A}(\mathbb{CP}_q^1))$, that is an infinite-dimensional free abelian group.

8 Monopoles and instantons on \mathbb{CP}_q^2

A review of the geometry of \mathbb{CP}_q^2 is in [20]. The full $*$ -calculus was given in [22]. We now review some results from [22] and [23] on monopoles and instantons as solutions of anti-self-duality equations.

8.1 The Hodge star operator on \mathbb{CP}_q^2

On a orientable Riemannian manifold M of (real) dimension n , there is a bimodule isomorphism $\Omega^k(M) \rightarrow \Omega^{n-k}(M)$ called the Hodge star operator: this is an isometry and has square ± 1 . It is usually defined in local coordinates, using the completely antisymmetric tensor and the determinant of the metric. With the Hodge star, one defines an inner product on the space of forms.

In the noncommutative case (lacking local coordinates), we proceed in the opposite way: we have a canonical Hermitian structure on forms, and we use this to define a map $*_H$ that we call “Hodge star operator”. We then show that on \mathbb{CP}_q^2 this has the correct properties and the correct $q \rightarrow 1$ limit.

The starting point to define $*_H$ is a differential $*$ -calculus $(\Omega^\bullet(\mathcal{A}), d)$ over a $*$ -algebra \mathcal{A} . To have a bimodule isomorphism $\Omega^k(\mathcal{A}) \rightarrow \Omega^{n-k}(\mathcal{A})$, for some n that we call “dimension” of the calculus, a necessary condition is that $\Omega^n(\mathcal{A})$ is a free \mathcal{A} -bimodule of rank 1, whose base element we denote by Φ . This is analogue to the condition that the space is orientable.

We also assume that each $\Omega^k(\mathcal{A})$, as a right module, has an Hermitian structure $(\cdot, \cdot) : \Omega^k(\mathcal{A}) \times \Omega^k(\mathcal{A}) \rightarrow \mathcal{A}$ and is self-dual³. Under this assumption, it is possible to prove there exists a right \mathcal{A} -module map $*_H : \Omega^k(\mathcal{A}) \rightarrow \Omega^{n-k}(\mathcal{A})$ uniquely defined by

$$(*_H \omega_1, \omega_2) \Phi = \omega_1^* \omega_2$$

for all $\omega_1 \in \Omega^k(\mathcal{A})$ and $\omega_2 \in \Omega^{n-k}(\mathcal{A})$ (the product on the right hand side is the product in $\Omega^\bullet(\mathcal{A})$). In particular, finitely generated projective modules with the canonical Hermitian structure are self-dual; in addition, for them it is possible to prove that the map $*_H$ is also a left \mathcal{A} -module map. More details on this topic will be reported in [23].

If $n = 4$, $\Omega^2(\mathcal{A})$ is the direct sum of the eigenspaces of $*_H$ corresponding to the eigenvalues $+1$ and -1 , called spaces of selfdual, respectively anti-selfdual (SD or ASD, for short) 2-forms.

On $\Omega^2(\mathbb{CP}_q^2)$ the Hodge star operator is given explicitly in [22, 23] in a way that we briefly describe. Similarly to the $q = 1$ case,

$$\Omega^{1,1}(\mathbb{CP}_q^2) = \Omega_0^{1,1}(\mathbb{CP}_q^2) \oplus (\Omega_0^{1,1}(\mathbb{CP}_q^2))^\perp$$

is the (orthogonal) direct sum of a rank 1 free $\mathcal{A}(\mathbb{CP}_q^2)$ -bimodule $\Omega_0^{1,1}(\mathbb{CP}_q^2)$ and its orthogonal complement. A basis element for $\Omega_0^{1,1}(\mathbb{CP}_q^2)$ is given by the $U_q(\mathfrak{su}(3))$ -invariant 2-form:

$$\Omega_q := \sum_{ijk} q^{2i} p_{ij} dp_{jk} \wedge_q dp_{ki} = \sum_{ij} q^{2i} \partial p_{ij} \wedge_q \bar{\partial} p_{ji} ,$$

where $p_{ij} = z_i^* z_j$ are the generators of $\mathcal{A}(\mathbb{CP}_q^2)$ (and we recall that one passes to the notations of [22, 23] with the replacement $z_i \rightarrow z_{3-i}$). For $q = 1$,

³ Amongst the many uses of this term, here we mean that the Hermitian structure yields also all homomorphisms of $\Omega^k(\mathcal{A})$, i.e. given any right \mathcal{A} -module homomorphism $\phi : \Omega^k(\mathcal{A}) \rightarrow \mathcal{A}$ there is $\eta \in \Omega^k(\mathcal{A})$ so that $\phi(\cdot) = (\eta, \cdot)$.

modulo a proportionality constant, this is just the Kähler form associated to the Fubini-Study metric [23].

There are two possible choices of orientation for \mathbb{CP}_q^2 , and the corresponding Hodge star operators differ by a sign. On \mathbb{CP}_q^2 with *standard orientation*, a 2-form is ASD if and only if it belongs to $(\Omega_0^{1,1}(\mathbb{CP}_q^2))^\perp$ (compare with the classical situation in [27]). On \mathbb{CP}_q^2 with *reversed orientation*, that we denote by $\overline{\mathbb{CP}_q^2}$, a 2-form is ASD if and only if it belongs to $\Omega^{0,2}(\mathbb{CP}_q^2) \oplus \Omega_0^{1,1}(\mathbb{CP}_q^2) \oplus \Omega^{2,0}(\mathbb{CP}_q^2)$; in particular, the Kähler form is ASD (for the classical situation compare with [28]).

8.2 ASD connections and Laplacians

Using the isomorphism $\Gamma_{-N} \simeq P_N \mathcal{A}(\mathbb{CP}_q^2)^{k_{N,2}}$ (with $k_{N,2} = \binom{|N|+2}{2}$) discussed in §3.2, one moves the Grassmannian connection of $\mathcal{E} = P_N \mathcal{A}(\mathbb{CP}_q^2)^{k_{N,2}}$ to Γ_{-N} . This yields a connection ∇_N given on $\eta \in \Gamma_{-N}$ by

$$\nabla_N \eta = \Psi_N^\dagger d(\Psi_N \eta) . \quad (33)$$

Its curvature is the operator of left multiplication by the 2-form ∇_N^2 in $\Omega^2(\mathbb{CP}_q^2)$ given by

$$\nabla_N^2 = \Psi_N^\dagger (dP_N)^2 \Psi_N . \quad (34)$$

In §3.3 we saw that Fredholm modules are a good replacement of Chern characters, as they are used to construct maps $K_0 \rightarrow \mathbb{Z}$ that are the analogue of characteristic classes (also called Chern-Connes characters in K-homology).

On the other hand on \mathbb{CP}_q^2 one can also mimic the construction of the usual Chern characters by associating to finitely generated projective modules (sections of noncommutative vector bundles) suitable integrals of powers of the Grassmannian connection (in fact of any connection). It appears that the correct framework for this is equivariant K-theory, as these integrals give numbers (that are not integer valued) depending only on the K-theory class of equivariant projective modules. These maps $K_0^{\mathcal{U}} \rightarrow \mathbb{R}$ are described in [22].

A connection on a bimodule will be called ASD if its curvature is a right-module endomorphism with coefficients in *anti-selfdual* 2-forms.

In [22] we studied $U(1)$ -monopoles on \mathbb{CP}_q^2 , i.e. ASD connections on the (line bundle) modules Γ_{-N} . The connection ∇_N on Γ_{-N} is the one in (33). The corresponding curvature ∇_N^2 , as in (34), is a scalar 2-form since right-module endomorphisms are given by $\text{End}_{\mathcal{A}(\mathbb{CP}_q^2)}(\Gamma_{-N}) \simeq \mathcal{A}(\mathbb{CP}_q^2)$. We showed

that ∇_N is left $U_q(\mathfrak{su}(3))$ -invariant, i.e. it commutes with the left action of $U_q(\mathfrak{su}(3))$. From this, it follows that the curvature is an invariant 2-form, and then it is ASD on \mathbb{CP}_q^2 . Explicitly, one has

$$\nabla_N^2 = q^{N-1}[N] \Omega_q .$$

In [23] we are continuing the project and describe $SU_q(2)$ one-instantons on \mathbb{CP}_q^2 , i.e. ASD connections on a ‘rank 2 homogeneous vector bundle’ with first Chern number equal to 0 and second Chern number equal to 1. Following Donaldson [28, Example 4.1.2] we choose the reverse orientation on \mathbb{CP}_q^2 . The ASD condition can be reformulated as a system of finite-difference equations (differential equations for $q = 1$, while derivatives are replaced by q -derivatives when $q \neq 1$), and provide a family of solutions ‘parametrized’ by a non-commutative space that is a cone over \mathbb{CP}_q^2 .

Given the monopole connection ∇_N on Γ_{-N} , one can also define the associated Laplacian $\Delta_N := (\nabla_N)^* \nabla_N$, where $(\nabla_N)^*$ is the adjoint of ∇_N . The eigenvalues $\{\lambda_{k,N}\}_{k \in \mathbb{N}}$ of Δ_N , explicitly computed in [22], are given by:

$$\begin{aligned} \lambda_{k,N} &= (1 + q^{-3})[k][k + N + 2] + [2][N] && \text{if } N \geq 0 , \\ \lambda_{k,N} &= (1 + q^{-3})[k + 2][k - N] + [2][N] && \text{if } N < 0 . \end{aligned}$$

We point out that for $q = 1$, $\lambda_{k,N} = 2(k^2 + kN + 2k + N) = \lambda_{k,-N}$ for any $N \geq 0$. On the other hand for $q \neq 1$, the spectrum of Δ_N is not symmetric under the exchange $N \leftrightarrow -N$; the quantization removes some degeneracies. A similar phenomenon was observed in [43] for \mathbb{CP}_q^1 . There is a simple relation,

$$\lambda_{k,N} - \lambda_{k,-N} = (1 - q^{-3})[2][N] , \quad \text{for all } N \geq 0 .$$

A On Chern characters and Fredholm modules

In Prop. (3.4) we gave maps

$$\varphi_k := \langle [F_k], \cdot \rangle : K_0(\mathcal{A}(\mathbb{CP}_q^n)) \rightarrow \mathbb{Z} ,$$

that, when $K_0(\mathcal{A}(\mathbb{CP}_q^n))$ is identified with \mathbb{Z}^{n+1} using the generators $[P_0]$, $[P_{-1}]$, ..., $[P_{-n}]$, are morphisms of abelian groups $\mathbb{Z}^{n+1} \rightarrow \mathbb{Z}$.

For $q = 1$, using the embeddings $\iota : \mathbb{CP}^k \rightarrow \mathbb{CP}^n$ one has has maps

$$\text{Ch}_k : K^0(\mathbb{CP}^n) \rightarrow \mathbb{Q} , \quad \text{Ch}_k(\mathcal{V}) = \int_{\mathbb{CP}^k} \iota^* \text{ch}_k(\mathcal{V}) ,$$

where $\mathcal{V} \rightarrow \mathbb{CP}^n$ is a vector bundle, and $\text{ch}_k(\mathcal{V})$ its k -th Chern character. Similarly to above, one can identify $K^0(\mathbb{CP}^n)$ with \mathbb{Z}^{n+1} using corresponding line bundles $L_0, L_{-1}, \dots, L_{-n}$, where $L_0 = \mathbb{CP}^n \times \mathbb{C}$ is the trivial line bundle, $L_{-1} \rightarrow \mathbb{CP}^n$ is the dual of the tautological bundle and $L_{-N} = (L_{-1})^{\otimes N}$. We compare the maps φ_k and Ch_k as morphisms of abelian groups $\mathbb{Z}^{n+1} \rightarrow \mathbb{Q}$. From Prop. (3.4) we know $\varphi_k(P_{-N}) = \binom{N}{k}$. We need to compute $\text{Ch}_k(L_{-N})$.

For a line bundle L , the total Chern character is $\text{ch}(L) = e^{c_1(L)}$, being the first Chern class $c_1(L)$ the only non-zero such a class for a line bundle. Since $\text{ch}(L \otimes L') = \text{ch}(L)\text{ch}(L')$, we have $\text{ch}(L_{-N}) = \text{ch}(L_{-1})^N = e^{Nc_1(L_{-1})}$ and $\text{ch}_k(L_{-N}) = \frac{N^k}{k!} c_1(L_{-1})^k$. By [30, Lemma 2.3.1], $x := \iota^* c_1(L_{-1})$ is exactly the first Chern number of the analogous bundle L_{-1} on \mathbb{CP}^k , and the integral is normalized such that $\int_{\mathbb{CP}^k} x^k = 1$. Therefore:

$$\text{Ch}_k(L_{-N}) = \frac{1}{k!} N^k = \frac{1}{k!} \sum_{j=0}^k j! \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \binom{N}{j},$$

where $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ are the Stirling numbers of the second kind [31]. Hence

$$\text{Ch}_k = \frac{1}{k!} \sum_{j=0}^k \left\{ \begin{matrix} k \\ j \end{matrix} \right\} j! \varphi_j,$$

as maps $\mathbb{Z}^{n+1} \rightarrow \mathbb{Q}$. In particular,

$$\text{Ch}_0 = \varphi_0, \quad \text{Ch}_1 = \varphi_1, \quad \text{Ch}_2 = \varphi_2 + \frac{1}{2} \varphi_1,$$

with their inverses: $\varphi_0 = \text{Ch}_0$, $\varphi_1 = \text{Ch}_1$ and $\varphi_2 = \text{Ch}_2 - \frac{1}{2} \text{Ch}_1$, the latter combination always being integer valued. These could be named the ‘rank’, ‘monopole number’ and ‘instanton number’ of the bundle, respectively.

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